Research Journal of Pure Algebra -2(6), 2012, Page: 167-175
Available online through www.rjpa.info ISSN 2248–9037

COMPARATIVE STUDY ON THE RATIOS OF NEVANLINNA'S CHARACTERISTIC OF TWO COMPOSITE MEROMORPHIC FUNCTIONS

Sanjib Kumar Datta[1*], Tanmay Biswas[2] and Soumen Kanti Deb[3]

[1]Department of Mathematics, University of Kalyani, Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India

[2]Rajbari, Rabindrapalli, R. N. Tagore Road, P. O. Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India

[3]Bahin High School, P. O. Bahin, Dist-Uttar Dinajpur, PIN-733157, West Bengal, India

(Received on: 21-04-12; Revised & Accepted on: 09-05-12)

ABSTRACT

In this paper we study the growth properties of composite entire and meromorphic functions which improve some earlier results of Lahiri and Datta [10] and Datta [5], [6].

AMS Subject Classification (2010): *30D35, 30D30.*

Keywords and Phrases: *Meromorphic function, entire function, composition, order(lower order), growth.*

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [7] and [11]. In the sequel we use the following notations:

$log^{[k]}x = log(log^{[k-1]}x)$ for $k = 1,2,3, \ldots$ and

$log^{[0]}x = x$; and

$exp^{[k]}x = exp(exp^{[k-1]}x)$ for $k = 1,2,3, \ldots$ and

$exp^{[0]}x = x$.

The following definitions are well known.

Definition 1: The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

If f is meromorphic then

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Definition 2: The type σ_f of a meromorphic function f is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When f is entire, it can be easily verified that

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \ 0 < \rho_f < \infty.$$

In this paper we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions which improve some earlier results.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] Let f be meromorphic and g be entire then for all sufficiently large values of r,

$$T(r, f\circ g) \leq \{1 + \circ(1)\} \frac{T(r,g)}{logM(r,g)} T(M(r,g), f).$$

Lemma 2: [2] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) \geq T(exp(r^{\mu}), f).$$

Lemma 3: [9] Let f be meromorphic and g be entire such that $0 < \mu < \rho_g \leq \infty$ and $\lambda_f > 0$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) \geq T(exp(r^{\mu}), g).$$

Lemma 4: [4] Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) < T(exp(r^{\mu}), f).$$

Lemma 5: [4] Let f be a meromorphic function of finite order and g be an entire function with $0 < \lambda_g < \mu < \infty$.

Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) < T(\exp(r^{\mu}), g).$$

Lemma 6: [8] Let f be an entire function of finite lower order. If there exists entire functions $d_i (i = 1,2, \ldots n; n \leq \infty$ satisfying $T(r, d_i) = o\{T(r,f)\}$ and $\sum_{i=1}^{n} \delta(d_i f) = 1$, then

$$\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = \frac{1}{\pi}.$$

3. THEOREMS

In this section we present the main results of the paper.

Theorem 1: Let f, h be two meromorphic functions and g , h be two entire functions such that $\rho_f < \infty$, $\rho_g < \rho_k$ and $\lambda_h > 0$ Then

$$\lim_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} = \infty.$$

Proof: As $\rho_g < \rho_k$ we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_g + \varepsilon < \rho_k - \varepsilon < \rho_k. \quad (1)$$

Since $T(r,g) \leq log^{+}M(r,g)$, we have from Lemma 1 for all sufficiently large values of r,

$$T(r, f\circ g) \leq \{1 + \circ(1)\} T(M(r,g), f)$$

i.e., $\log T(r, f\circ g) \le \left(\rho_f + \varepsilon\right) \log M(r, g) + O(1)$

i.e., $\log T(r, f\circ g) \le \left(\rho_f + \varepsilon\right) r^{(\rho_g+\varepsilon)} + O(1)$. (2)

Again from (1) and Lemma 2 it follows for a sequence of values of r tending to infinity that

$\log T(r, h\circ k) \ge \log T\left(exp r^{(\rho_k-\varepsilon)}, h\right)$

i.e., $\log T(r, h\circ k) \ge (\lambda_h - \varepsilon) \log exp\, r^{(\rho_k-\varepsilon)}$

i.e., $\log T(r, h\circ k) \ge (\lambda_h - \varepsilon)\, r^{(\rho_k-\varepsilon)}$. (3)

Therefore from (2) and (3) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T(r,h\circ k)}{\log T(r,f\circ g)} \ge \frac{(\lambda_h-\varepsilon)\, r^{(\rho_k-\varepsilon)}}{(\rho_f+\varepsilon) r^{(\rho_g+\varepsilon)}+O(1)}. \quad (4)$$

Now in view of (1) it follows from (4) that

$$\lim_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} = \infty.$$

This proves the theorem.

Remark 1: For the validity of the Theorem 1, the conditions $\lambda_h > 0$, $\rho_g < \rho_k$ and $\rho_f < \infty$ are necessary but for meromorphic f with order zero Theorem 1 also holds for $\rho_g \ge \rho_k$ which are evident from the following examples.

Example 1: Let $f = g = exp\, z, h = z$ and $k = expz^2$.

Then $\rho_f = \rho_g = 1, \lambda_h = \rho_h = 0$ and $\lambda_k = \rho_k = 2$.

Now
$T(r, h\circ k) \le \log M(r, h\circ k) = \log exp(r)^2 = r^2 and\ T(r, f\circ g) \sim = \frac{expr}{(2\pi^3 r)^{\frac{1}{2}}}$.
So
$$\frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} \le \frac{2\log r}{r - \frac{1}{2}\log r + O(1)}$$

$$i.e., \quad \limsup_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} = 0.$$

Example 2: Suppose $f = h = k = exp\, z$, and $g = expz^2$.

Then $\rho_f = \lambda_h = \rho_h = \lambda_k = \rho_k = 1$ and $\rho_g = 2$.

Now
$3T(2r, f\circ g) \ge \log M(r, f\circ g) = exp(r)^2$
$i.e., \ \log T(r, f\circ g) \ge \frac{r^2}{4} + O(1)$.

Also
$$T(r, h\circ k) \sim = \frac{expr}{(2\pi^3 r)^{\frac{1}{2}}}.$$
Therefore
$$\frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} \le \frac{r - \frac{1}{2}\log r + O(1)}{\frac{r^2}{4} + O(1)}$$

$$i.e., \quad \limsup_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} = 0.$$

Example 3: Let $f = \exp^{[2]} z,\ g = h = \exp z$ and $k = \exp z^2$.

Then $\rho_f = \infty,\ \lambda_h = \rho_g = 1 < 2 = \lambda_k = \rho_k$.

Now
$T(r, h\circ k) \le \log M(r, h\circ k) = \exp(r^2)$ and $3T(2r, f\circ g) \ge \log M(r, f\circ g) = \exp^{[2]} r$.

So
$$\frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} \le \frac{r^2}{\exp\frac{r}{2} + O(1)}$$

$$i.e., \quad \limsup_{r\to\infty} \frac{T(r, h\circ k)}{T(r, f\circ g)} = 0\,.$$

Example 4: Suppose $f = z,\ g = \exp z^2$ *and* $h = k = \exp z$.

Then $\lambda_f = \rho_f = 0 < \infty,\ \lambda_h = \rho_h = \lambda_k = \rho_k = 1 < 2 = \rho_g$.

Therefore
$T(r, f\circ g) \le \log M(r, f\circ g) = r^2$

$i.e., \quad \log T(r, f\circ g) \le 2\log r.$

And also
$$T(r, h\circ k) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\ (r\to\infty).$$

Thus
$$\frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} \ge \frac{r - \frac{1}{2}\log r + O(1)}{2\log r}$$

$$i.e., \quad \limsup_{r\to\infty} \frac{T(r, h\circ k)}{T(r, f\circ g)} = \infty\,.$$

Example 5: Suppose $f = z,\ g = h = k = \exp z$.

Then $\lambda_f = \rho_f = 0 < \infty,\ \lambda_h = \rho_h = \lambda_k = \rho_k = 1 = \rho_g$.

Therefore
$T(r, f\circ g) \le \log M(r, f\circ g) = r$

$i.e., \quad \log T(r, f\circ g) \le \log r.$

And
$$T(r, h\circ k) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\ (r\to\infty).$$

Thus
$$\frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} \ge \frac{r - \frac{1}{2}\log r + O(1)}{\log r}$$

$$i.e., \quad \limsup_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} = \infty\,.$$

Corollary 1: Under the assumptions of Theorem 1,

$$\limsup_{r\to\infty} \frac{T(r, h\circ k)}{T(r, f\circ g)} = \infty .$$

Proof: By Theorem 1 we have for a sequence of values of r tending to infinity and for $K > 1$,

$\log T(r, h\circ k) \geq K \log T(r, h\circ k)$

$i.e., \ T(r, f\circ g) > \{\log T(r, h\circ k)\}^K,$

from which the corollary follows.

In the line of Theorem 1 one can easily prove the following theorem:

Theorem 2: Let $f, \ h$ be two meromorphic functions and g, k be two entire functions such that $\lambda_h > 0, \ \rho_k > 0$, $\rho_f < \infty$, and $\rho_g < \infty$.Then

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, h\circ k)}{\log^{[2]} T(r, f\circ g)} \geq \frac{\rho_k}{\rho_g} .$$

The proof is omitted.

In the line of Theorem 2 the following corollary may be deduced:

Corollary 2: Let $f, \ h$ be two meromorphic functions and g, k be two entire functions such that $\lambda_h > 0, \ \rho_k > 0$, $\rho_f < \infty$, and $\rho_g < \infty$.Then

$$\limsup_{r\to\infty} \frac{\log^{[3]} T(r, h\circ k)}{\log^{[3]} T(r, f\circ g)} \geq 1 .$$

Remark 2: Under the same condition of Corollary 2, one can easily verify that

$$\limsup_{r\to\infty} \frac{\log^{[3]} T(r, h\circ k)}{\log^{[3]} T(r, f\circ k)} \geq 1 .$$

Theorem 3: Let f and h be meromorphic and g and k be entire such that (i) $\rho_f < \infty$, (ii) $\lambda_h > 0$, (iii) $\lambda_k > 0$, (iv) $\rho_g < \rho_k$ and (v) $0 < \rho_g < \infty, \ \sigma_g < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r, h\circ k)} \leq \min\left\{\frac{\rho_f \sigma_g}{\lambda_h}, \frac{\rho_f \sigma_g}{\lambda_k}\right\}.$$

Proof: Since $T(r, g) \leq \log^+ M(r, g)$, by Lemma 1 we obtain for all sufficiently large values of r,

$T(r, f\circ g) \leq \{1 + o(1)\} T(M(r, g), f)$

$i.e., \ \log T(r, f\circ g) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1)$

$i.e., \ \log T(r, f\circ g) \leq (\rho_f + \varepsilon)(\sigma_g + \varepsilon)\, r^{\rho_g} + O(1).$ (5)

Since $\rho_g < \rho_k$, in view of Lemma 2 it follows for a sequence of values of r tending to infinity that

$\log T(r, h\circ k) \geq \log T(\exp(r^{\rho_g}), h)$

$i.e., \ \log T(r, h\circ k) \geq (\lambda_h - \varepsilon) \log\exp(r^{\rho_g})$

$i.e., \ \log T(r, h\circ k) \geq (\lambda_h - \varepsilon) r^{\rho_g}.$ (6)

Similarly in view of Lemma 3 we have for a sequence of values of r tending to infinity

$$\log T(r, h \circ k) \geq \log T(\exp(r^{\rho_g}), k)$$

$$i.e., \ \log T(r, h \circ k) \geq (\lambda_k - \varepsilon) \log \exp(r^{\rho_g})$$

$$i.e., \ \log T(r, h \circ k) \geq (\lambda_k - \varepsilon) r^{\rho_g}, \tag{7}$$

where $0 < \varepsilon < \min\{\lambda_h, \lambda_k\}$.

Now from (5) and (6) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T(r, f \circ g)}{\log T(r, h \circ k)} \leq \frac{(\rho_f + \varepsilon)(\sigma_g + \varepsilon) r^{\rho_g} + O(1)}{(\lambda_h - \varepsilon) r^{\rho_g}}$$

$$i.e., \ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, h \circ k)} \leq \frac{\rho_f \sigma_g}{\lambda_h}. \tag{8}$$

Analogously from (5) and (7) it follows for a sequence of values of r tending to infinity that

$$\frac{\log T(r, f \circ g)}{\log T(r, h \circ k)} \leq \frac{(\rho_f + \varepsilon)(\sigma_g + \varepsilon) r^{\rho_g} + O(1)}{(\lambda_k - \varepsilon) r^{\rho_g}}$$

$$i.e., \ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, h \circ k)} \leq \frac{\rho_f \sigma_g}{\lambda_k}. \tag{9}$$

Thus the theorem follows from (8) and (9):

In the line of Theorem 3 one can easily prove the following theorem:

Theorem 4: Let f and h be meromorphic and g and k be entire such that (i) $\rho_h < \infty$, (ii) $\lambda_f > 0$, (iii) $\lambda_g > 0$, (iv) $\rho_k < \rho_g$ and (v) $0 < \rho_k < \infty$, $\sigma_k < \infty$. Then

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, h \circ k)} \geq \max\left\{\frac{\lambda_f}{\rho_h \sigma_k}, \frac{\lambda_g}{\rho_h \sigma_k}\right\}.$$

The proof is omitted.

Theorem 5: Let f be a meromorphic function and g, h, k be three entire functions such that $\lambda_f > 0, \lambda_g > 0$ and $0 < \rho_{h \circ k} < \rho_g$. Also let $a_i (i = 1, 2, \ldots, n;\ n \leq \infty)$ be entire functions such that

(i) $T(r, a_i) = o\{T(r, h \circ k)\}$ and (ii) $\sum_{i=1}^{n} \delta(a_i; h \circ k) = 1$. Then

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, h \circ k)} \geq \max\{\pi \lambda_f, \pi \lambda_g\}.$$

Proof: Let us choose $\varepsilon\ (> 0)$ such that

$$\rho_{h \circ k} + \varepsilon < \rho_g - \varepsilon < \rho_g. \tag{10}$$

Now in view of (10) and Lemma 2 we obtain for a sequence of values of r tending to infinity that

$$\log T(r, f \circ g) \geq \log T(\exp(r^{\rho_{h \circ k} + \varepsilon}), f)$$

$$i.e., \ \log T(r, f \circ g) \geq (\lambda_f - \varepsilon) \log \exp(r^{\rho_{h \circ k} + \varepsilon})$$

$$i.e., \ \log T(r, f \circ g) \geq (\lambda_f - \varepsilon) r^{\rho_{h \circ k} + \varepsilon}. \tag{11}$$

Again from the definition of order, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} M(r, h\circ k) \leq (\rho_{h\circ k} + \varepsilon)\log r$$

$$i.e.,\ \log M(r, h\circ k) \leq r^{\rho_{h\circ k}+\varepsilon}. \quad (12)$$

Combining (11) and (12) we get for a sequence of values of r tending to infinity,

$$\log T(r, f\circ g) \geq (\lambda_f - \varepsilon)\log M(r, h\circ k)$$

$$\text{i.e., } \frac{\log T(r, f\circ g)}{T(r, h\circ k)} \geq (\lambda_f - \varepsilon)\frac{\log M(r, h\circ k)}{T(r, h\circ k)}$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} \geq (\lambda_f - \varepsilon)\liminf_{r\to\infty}\frac{\log M(r, h\circ k)}{T(r, h\circ k)}.$$

As $\varepsilon\ (>0)$ is arbitrary, with the help of Lemma 6 it follows from above that

$$\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} \geq \pi\,\lambda_f. \quad (13)$$

Also in view of (10) and Lemma 3 we have for a sequence of values of r tending to infinity

$$\log T(r, f\circ g) \geq \log T(\exp(r^{\rho_{h\circ k}+\varepsilon}), g)$$

$$i.e.,\ \log T(r, f\circ g) \geq (\lambda_g - \varepsilon)\log\exp(r^{\rho_{h\circ k}+\varepsilon})$$

$$i.e.,\ \log T(r, f\circ g) \geq (\lambda_g - \varepsilon) r^{\rho_{h\circ k}+\varepsilon}. \quad (14)$$

Now from (12) and (14) we get for a sequence of values of r tending to infinity,

$$\log T(r, f\circ g) \geq (\lambda_g - \varepsilon)\log M(r, h\circ k)$$

$$\text{i.e., } \frac{\log T(r, f\circ g)}{T(r, h\circ k)} \geq (\lambda_g - \varepsilon)\frac{\log M(r, h\circ k)}{T(r, h\circ k)}$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} \geq (\lambda_g - \varepsilon)\liminf_{r\to\infty}\frac{\log M(r, h\circ k)}{T(r, h\circ k)}. \quad (15)$$

Since $\varepsilon\ (>0)$ is arbitrary, in view of Lemma 6 we get from above that

$$\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} \geq \pi\,\lambda_g. \quad (16)$$

Thus the theorem follows from (13) and (16).

Theorem 6: Let f be a meromorphic function and g, h, k be three entire functions such that (i) $\lambda_g < \lambda_{h\circ k} < \infty$ (ii) $0 < \lambda_f \leq \rho_f < \infty$ and (iii) $\rho_g < \infty$. Also let $a_i (i = 1, 2, \ldots, n;\ n \leq \infty)$ be entire functions such that (i) $T(r, a_i) = o\{T(r, h\circ k)\}$ and (ii) $\sum_{i=1}^{n} \delta(a_i; h\circ k) = 1$. Then

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} \leq \min\{\pi\,\rho_f, \pi\,\rho_g\}.$$

Proof: Let us choose $\varepsilon\ (>0)$ such that

$$\lambda_g < \lambda_g + \varepsilon < \lambda_{h\circ k} - \varepsilon. \quad (17)$$

Now in view of (17) and Lemma 4 we obtain for a sequence of values of r tending to infinity that

$$\log T(r, f\circ g) < \log T\big(\exp(r^{\lambda_{h\circ k}-\varepsilon}), f\big)$$

$$i.e.,\ \log T(r, f\circ g) < (\rho_f + \varepsilon)\log\exp(r^{\lambda_{h\circ k}-\varepsilon})$$

$$i.e.,\ \log T(r, f\circ g) < (\rho_f + \varepsilon)\, r^{\lambda_{h\circ k}-\varepsilon}. \qquad (18)$$

Again from the definition of order, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} M(r, h\circ k) \geq (\lambda_{h\circ k} - \varepsilon)\log r$$

$$i.e.,\ \log M(r, h\circ k) \geq r^{\lambda_{h\circ k}-\varepsilon}. \qquad (19)$$

Combining (18) and (19) we get for a sequence of values of r tending to infinity,

$$\log T(r, f\circ g) < (\rho_f + \varepsilon)\log M(r, h\circ k)$$

$$\text{i.e., } \frac{\log T(r, f\circ g)}{T(r, h\circ k)} < (\rho_f + \varepsilon)\frac{\log M(r, h\circ k)}{T(r, h\circ k)}$$

$$\text{i.e., } \liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} < (\rho_f + \varepsilon)\limsup_{r\to\infty}\frac{\log M(r, h\circ k)}{T(r, h\circ k)}$$

As $\varepsilon\ (> 0)$ is arbitrary, with the help of Lemma 6 it follows from above that

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} \leq \pi\,\rho_f. \qquad (20)$$

Also in view of Lemma 5 and (17) we have for a sequence of values of r tending to infinity

$$\log T(r, f\circ g) < \log T\big(\exp(r^{\lambda_{h\circ k}-\varepsilon}), g\big)$$

$$i.e.,\ \log T(r, f\circ g) < (\rho_g + \varepsilon)\log\exp(r^{\lambda_{h\circ k}-\varepsilon})$$

$$i.e.,\ \log T(r, f\circ g) < (\rho_g + \varepsilon)\, r^{\lambda_{h\circ k}-\varepsilon}. \qquad (21)$$

Now from (19) and (21) we get for a sequence of values of r tending to infinity,

$$\log T(r, f\circ g) < (\rho_g + \varepsilon)\log M(r, h\circ k)$$

$$\text{i.e., } \frac{\log T(r, f\circ g)}{T(r, h\circ k)} < (\rho_g + \varepsilon)\frac{\log M(r, h\circ k)}{T(r, h\circ k)}$$

$$\text{i.e., } \liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} < (\rho_g + \varepsilon)\limsup_{r\to\infty}\frac{\log M(r, h\circ k)}{T(r, h\circ k)}.$$

Since $\varepsilon\ (> 0)$ is arbitrary, in view of Lemma 6 we get from above that

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, h\circ k)} \leq \pi\,\rho_g. \qquad (22)$$

Thus the theorem follows from (20) and (22).

The following theorem is a natural consequence of Theorem 5 and Theorem 6.

Theorem 7: Let f be a meromorphic function with regular growth i.e., $\rho_f = \lambda_f$ and g, h, k be three entire functions such that (i) $0 < \lambda_g < \lambda_{h\circ k} \leq \rho_{h\circ k} < \rho_g < \infty$ *and* (ii) $0 < \rho_f < \infty$. Also let $a_i (i = 1, 2, \ldots, n;\ n \leq \infty)$ be entire functions such that (i) $T(r, a_i) = o\{T(r, h\circ k)\}$ and (ii) $\sum_{i=1}^{n} \delta(a_i; h\circ k) = 1$. Then

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,h\circ k)}\leq \pi\,\rho_f\leq \limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,h\circ k)}.$$

The proof is omitted.

4. OPEN PROBLEM

One may generalize the above growth estimations of composite entire and meromorphic functions on the basis of $(p,q)-th$ order ($(p,q)-th$ lower order) the idea of which have already been developed by Datta and Biswas in [3].

5. REFERENCES

[1] Bergweiler, W: On the Nevanlinna Characteristic of a composite function, Complex Variables, Vol. 10(1988), pp.225-236.

[2] Bergweiler, W: On the growth rate of composite meromorphic functions, Complex Variables, Vol. 14(1990), pp.187-196.

[3] Datta, S.K. and Biswas, T. : On the growth estimate of composite entire and meromorphic functions, Bulletin of Mathematical Analysis and Applications, Vol. 2 No. 2(2010), pp.1-17.

[4] Datta, S.K. and Biswas, T. : On a result of Bergweiler, Int. J. Pure Appl. Math., Vol.51, No.1 (2009), pp.33-37.

[5] Datta, S.K. : Some results on the growth properties of composition of entire and meromorphic functions, Bull. Cal. Math. Soc., Vol. 96, No. 5(2004), pp. 351-360.

[6] Datta, S.K. : Composite meromorphic and entire functions and their growth properties, Journal of Indian Acad. Math., Vol 26, No2(2004), pp. 351-363.

[7] Hayman, W.K.: Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[8] Lin Q. and Dai C. : On a conjecture of Shah concerning small functions, Kexue Tong bao, Vol. 31, No 4(English Ed.)(1986), pp. 220-224.

[9] Lahiri, I. and Sharma, D.K. : Growth of composite entire and meromorphic functions, Indian J. Pure Appl. Math., Vol. 26, No. 5(1995), pp.451-458.

[10] Lahiri, I. and Datta, S. K.: On the growth properties of composite entire and meromorphic functions, Bull. Allahabad Math. Soc., Vol, 18(2003), pp.15-34.

[11] Valiron, G. : Lectures on the General Theory of Integral Functions, Chelsea Publishing Company, 1949.

Source of support: Nil, Conflict of interest: None Declared