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# STUDY OF JACOBSON RADICAL OF THE GROUP ALGEBRA OF A GROUP IS EITHER COMMUTATIVE OR CENTRAL SUB ALGEBRA

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**ABSTRACT**

*This study constructed the relationship between Radical and Group Algebra. In order to construct the relationship between radical and group algebra let a group algebra K (G) of a finite group G over the field K of characteristic $p \neq 0$ has a nonzero radical R if and only if p is a divisor of o(G), the order of G. This study shows that the appearances were not deceptive in the problem of centrality, for odd primes and in the problem of commutativity. Finally after introducing commutativity of Ring Theory and using some Theorems and Lemmas we have verified that Jacobson radical of the Group Algebra of a Group is either commutative or a central sub algebra.*

***Key definition:*** *Group, Finite group, Subgroup, Abelian group, Commutator, Sylow p-subgroup, Homomorphism, Isomorphism.*

## INTRODUCTION

This study discussed preliminaries and some basic important and useful definitions with examples. All these definitions are required as prerequisite to carry on research work and also have discussed about Radical Theory, Equivalent characteristic and Properties of Jacobson radical. Then giving some ideas about development of Group Theory have discussed the Normal subgroup, some properties of Normal subgroup, Sylow *p*-subgroup, Solvable group and Torsion Free group and proof their related theorems with some applications.

In order to construct the relationship between radical and group algebra suggests that R is related to the Sylow *p*-groups of G and so R is define in terms of these subgroups. As a generalization of this define $R'$ to be the intersection of all the left ideals of K(G) generated by the radicals of the group algebras of the Sylow *p*-groups of G. Then $R'$ is a nilpotent ideal of K(G), and Lombardo-Radici has shown that $R'$ = R and for that reason have shown that $R'$ = R if one of the following conditions is satisfied:

(A) G is homomorphic with a Sylow *p*-group of G.
(B) G is a super-solvable group.
(C) G is a solvable group with (o(G), $p^2$) = *p*.

Let R be a ring with center Z, Jacobson radical J, and set N of all nilpotent elements. Call R semiperiodic if for each $x \in$ R (J $\cup$ Z), there exist positive integer *m, n* of opposite parity such that $x^n - x^m \in N$. These studies have investigated commutativity of semiperiodic rings, and have provided noncommutative examples.

This study have studied the situation as to when the unite group U(KG) of a group algebra KG equals K*G(1+J(KG)), where K is a field of characteristic $p > 0$ and G is a finite group. Here have highlighted some basic things so that it can move to verify that Jacobson Radical of the Group Algebra of a Group is either Commutative or a central subalgebra.

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The earlier formulations of study results are not quite suitable for present purposes. According certain further conclusions, which were derived in the course of the original proofs, and incorporating some significant additions for which justification is subsequently given.

To shown the final result, this study have considered a fixed algebraically closed field K of characteristic $p > 0$. The twisted group of a group G over K is denoted by $K^t(G)$, the ordinary group algebra being denoted simply by K(G), the corresponding Jacobson radicals are denoted by $JK^t(G)$ and JK(G) respectively. If A is a subgroup of G, |A| is the order of A and $C_G$(A) is the centralizer of A in G. If B is also a subgroup of G, [A. B] is the subgroup generated by the elements $a^{-1}b^{-1}ab$ ($a \in$A, $b \in$ B), in particular $G'$ = [$G$, $G$] and $G''$ = [$G'$, $G'$] are the successive derived groups of G. $P$ denotes a $p$-Sylow subgroup of G and {e} is the trivial group.

**PRELIMINARIES AND SOME DEFINITION**

**Group:** A group G is a finite or infinite set of elements together with a binary operation (called the group operation) that together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. The operation with respect to which a group is defined is often called the "group operation," and a set is said to be a group "under" this operation. Elements, A, B, C, …with binary operation between A and B denoted AB form a group if

1. Closure: If A and B are two elements in G, then the product A,B is also in G.
**2.** Associativity: The defined multiplication is associative, i.e., for all, A, B, C $\in$ G, (AB)C=A(BC).

**3.** Identity: There is an identity element I(a.k.a. 1, E, or e) such that IA=AI=A for every element A$\in$ G.

**4.** Inverse: There must be an inverse (a.k.a. reciprocal) of each element. Therefore, for each element A of G, the set contains an element B=$A^{-1}$such that A $A^{-1}$ = $A^{-1}$ A=I.

**Finite group:** group G with a finite number of elements is called a finite group; otherwise it is called an infinite group.

**Subgroup:** Let (G,*) be a group. A non-empty subset H of G is called a subgroup of G if the restrictions of the operation on G to H makes (G,*) a group in its own right.

**Abelian group:** Let (G,*) be a group. If $a*b = b*a$, for all $a, b \in G$. We call G is abelian or commutative group. If here exists a pair of elements $a, b \in G$ such that $a*b \neq b*a$. we say that G is non-commutative or non-abelian. When a*b = b*a, we say that the elements *a* and *b* commute. .

**Commutator:** Let a be a group and $a,b \in G$. Then, the element $aba^{-1}b^{-1}$ is called the comutator of (*a*,*b*).The smallest subgroup containing$\{aba^{-1}b^{-1}: a, b \in G\}$ is called the commutator subgroup of G .

**Isomorphism:** If the homomorphism f is one-one and onto then f is called an isomorphism ,i,e. a homomorphism $f$:G $\rightarrow$ H is called an isomorphism if *f* is bijective.

**Sylow *p*-subgroup:** Let G be a finite group of order $p^m n$, where *p* is prime and *p* is not a divisor of *n*. Then, a **subgroup H of G is said to be a sylow *p*-subgroup, if o(H) is the highest power of *p* that divides o(G).**

**Homomorphism:** Let (G,∘) be a group and (H,*) be another group. Let f be the function f:G $\rightarrow$H from G to H such that $f(a \circ b) = f(a)*f(b)$, $\forall a, b \in G$.

Then *f* is called a homomorphism.

**Normal sub-groups:** A subgroup H of a group G is said to be a normal sub-group of G if H$a$= $a$H $\forall\ a \in G$. A normal sub-group is also known as invariant sub-group or a self conjugate sub-group or a normal divisor of the group.

 

For a group G, G and {e} are always normal sub-group of G and these are called trivial normal sub-groups.

**Some properties of Normal sub-group, Sylow p-subgroup, Solvable group and Torsion free group**

A subgroup, N, of a group, G, is called a normal subgroup , if it is invariant under conjugation; that is, for each element

*n* in N and each g in G, the element $gng^{-1}$ is still in N. We write $N \triangleleft G \Leftrightarrow \forall\, n \in N, \forall\, g \in G, gng^{-1} \in N$

For any subgroup, the following conditions are equivalent to normality. Therefore any one of them may be taken as the definition

- For all g in G, $gNg^{-1} \subseteq N$.
- For all g in G, $gNg^{-1} = N$.
- The sets of left and right cosets of N in G coincide.
- For all g in G, gN = Ng.
- N is a union of conjugacy classes of G.
- There is some homomorphism on G for which N is the kernel.

The last condition accounts for some of the importance of normal subgroups; they are a way to internally classify all homomorphisms defined on a group. For example, a non-identity finite group is simple if and only if it is isomorphic to all of its non-identity homomorphic images, a finite group is perfect if and only if it has no normal subgroups of prime index, and a group is imperfect if and only if the derived subgroup is not supplemented by any proper normal subgroup.

**SOME PROPERTIES OF NORMAL SUB-GROUPS**

**(A).** Normality is preserved upon subjective homomorphisms, and is also preserved upon taking inverse images.

**(B).** Normality is preserved on taking direct products.

**(C).** A normal subgroup of a normal subgroup of a group need not be normal in the group. That is, normality is not a transitive relation. However, a characteristic subgroup of a normal subgroup is normal. Also, a normal subgroup of a central factor is normal. In particular, a normal subgroup of a direct factor is normal.

**(D).** Every subgroup of index 2 is normal. More generally, a subgroup H of finite index *n* in G contains a subgroup K normal in G and of index dividing *n*! called the normal core. In particular, if *p* is the smallest prime dividing the order of G, then every subgroup of index *p* is normal

**Proposition:** For $H \leq G$, the following are equivalent:

**(i)** $H \triangleleft G$

**(ii)** for every $a \in G$, $aHa^{-1} = H$

***(iii)*** for every $a \in G$, $h \in H$, $aha^{-1} \in H$. That is, if $h \in H$, then all conjugates of *h* are also in H.

**Sylow p-subgroup:** For a prime number *p*, a Sylow *p*-subgroup (sometimes *p*-Sylow subgroup) of a group G is a maximal *p*-subgroup of G, i.e., a subgroup of G which is a *p*-group (so that the order of any group element is a power of *p*), and which is not a proper subgroup of any other *p*-subgroup of G. The set of all Sylow *p*-subgroups for a given prime *p* is sometimes written $Syl_p(G)$.

**Sylow theorems:** In mathematics, specifically in the field of finite group theory, the Sylow theorems are a collection of theorems named after the Norwegian mathematician L. Sylow (1872) that give detailed information about the number of subgroups of fixed order a given finite group contains. The Sylow theorems form a fundamental part of finite group theory and have very important applications in the classification of finite simple groups.

The Sylow theorems assert a partial converse to Lagrange's theorem that for any finite group G the order (number of elements) of every subgroup of G divides the order of G: For any prime power $p^n$ which divides the order of a given finite group G, there exists a subgroup of G of order $p^n$. For any prime factor *p* of the order of a finite group G, there exists a Sylow *p*-subgroup of G. The order of a Sylow *p*-subgroup of a finite group G is $p^n$, where *n* is the multiplicity of *p* in the order of G, and any subgroup of order $p^n$ is a Sylow *p*-subgroup of G. The Sylow *p*-subgroups of a group (for fixed prime *p*) are conjugate to each other. The number of Sylow *p*-subgroups of a group for fixed prime *p* is congruent to 1 mod *p*.

Collections of subgroups which are each maximal in one sense or another are common in group theory. The surprising result here is that in the case of $\mathrm{Syl}_p(G)$, all members are actually isomorphic to each other and have the largest possible order: if $|G| = p^n m$ with $n > 0$ where $p$ does not divide $m$, then any Sylow $p$-subgroup $P$ has order $|P| = p^n$. That is, $P$ is a $p$-group and $\gcd(|G:P|, p) = 1$. These properties can be exploited to further analyze the structure of G.

The following theorems were first proposed and proven by Ludwig Sylow in 1872, and published in Mathematische Annalen.

**TORSION FREE GROUP**

Let G be a group. An element g of G is called a torsion element if g has finite order. If all elements of G are torsion, then G is called a torsion group. If the only torsion element is the identity element, then the group G is called torsion-free.

Let M be a module over a ring R without zero divisors. An element m of M is called a torsion element if the cyclic submodule of M generated by m is not free. Equivalently, m is torsion if and only if it has a non-zero annihilator in R. If the ring R is an integral domain, then the set of all torsion elements forms a submodule of M, called the torsion submodule of M, sometimes denoted T(M). The module M is called a torsion module if T(M) = M, and is called torsion-free if T(M) = 0. Note that when R is only a commutative ring, torsion elements of M might not form a submodule. If the ring R is non-commutative then the situation is more complicated, and the set of torsion elements need not be a submodule. Nevertheless, it is a submodule given the assumption that the ring R satisfies the Ore condition. This covers the case when R is a Noetherian domain. Any abelian group may be viewed as a module over the ring $\mathbb{Z}$ of integers, and in this case the two notions of torsion coincide.

More generally, let R be an arbitrary ring and S $\subset$ R be a multiplicatively closed subset. Then one defines the notion of

S-torsion as follows. An element m of an R-module M is called an S-torsion element if there exists s in S such that *s* annihilates *m*, i.e., $sm = 0$. In particular, one can take for S to be the set of all non-zero divisors of the ring R. In this case, S-torsion is frequently called simply torsion, extending the definition above from the case of domains to general rings.

**ON THE RADICAL OF A GROUP ALGEBRA**

A basic result in the study of group algebras and characters states that the group algebra $\mathrm{K}$(G) of a finite group G over the field $\mathrm{K}$ of characteristic p $\neq 0$ has a nonzero radical R if and only if $p$ is a divisor of o(G), the order of G. This suggests that R is related in some manner to the Sylow $p$-groups of G and that it may be possible to define R in terms of these subgroups. we know that if o(G)= $p^a$, then R is of dimension $p^a - 1$ and has as a basis the set of elements $P_i - 1$. As a generalization of this define R' to be the intersection of all the left ideals of K(G) generated by the radicals of the group algebras of the Sylow $p$-groups of G. Then R' is a nilpotent ideal of K(G), and Lombardo-Radici has shown that R' =R provided G has a unique Sylow p-group or o(G) = $pq$ where $q$ is also a prime. Also, know that if G is the simple group of order 60 and if $p$ = 2 or 3 then R' is a proper subideal of R.

In this chapter it will be shown that R' = R if one of the following conditions is satisfied:

**(A)** G is homomorphic with a Sylow $p$-group of G.
**(B)** G is a super-solvable group.
**(C)** G is a solvable group with (o (G), $p^2$) = $p$.

In the last section of this chapter an application to a related problem is made. If G contains an invariant $p$-group then K(G) is bound to its radical R (i.e., if $a$ in K(G) is an element such that $a$R = R$a$ = 0, then $a$ is in R). This raises the question : If K(G) is bound to its radical R, does G contain an invariant $p$-group ? This is equivalent to the question: Does G contain an invariant $p$-group if G possesses no irreducible representation of highest kind? (An irreducible representation of highest kind is one whose dimension is divisible by the highest power of $p$ which divides o(G).) It is shown that if G is a group such that R' = R and if the Sylow p-groups of G are cyclic, then the above question is answered affirmatively. Also an example is given where the answer is negative.

The radical R of the group algebra K(G) of the group G over the field K equals R', the intersection of all the left ideals of K(G) generated by the radicals of the group algebras of the Sylow $p$-groups of G.

The radical R of the group algebra K(G) of a supersolvable group G over the field K equals R'.

The radical R of the group algebra K(G) of the group G over the field K equals R'.

If the group G contains an invariant *p*-subgroup $\mathcal{P}$, then the group algebra K(G) of G over a field of characteristic *p* is a bound algebra.

If the Sylow *p*-groups of G are cyclic and if the radical R of K(G) equals R' then G contains an invariant *p*-subgroup if' K(G) is bound to R.

**ON THE COMMUTATIVITY OF SEMIPERIODIC RINGS**

Let R be a ring with center Z = Z(R), Jacobson radical J = J(R), and set N = N(R) of all nilpotent elements; and let $\mathbb{Z}$ and $\mathbb{Z}^+$denote the ring of integers and the set of positive integers. Define R to be periodic if for each *x* ∈ R, there exist distinct positive integers *m*, *n* such that $x^n - x^m$. It is known that R must be periodic if each *x* ∈ R satisfies the Chacron criterion:

There exists *m* ∈ $\mathbb{Z}^+$and *p*(*t*) ∈ Z[*t*] such that $x^m - x^{m+1}p(x)$ .

It follows that R is periodic if for each *x* ∈ R there exist distinct *m, n* ∈ $\mathbb{Z}^+$ for which $x^n - x^m$∈ N.

In this study rings in which an appropriate subset of elements of R satisfy the Chacron criterion. Specifically, we define R to be semiperiodic if for each *x* ∈ R \ (J ∪ Z) there exist *m*, *n* ∈$\mathbb{Z}^+$, of opposite parity, such that $x^n - x^m$∈ N. Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain non-nil periodic rings; and it contains the generalized periodic-like rings. We shall be principally concerned with commutativity and near-commutativity of semiperiodic rings. Here begin with a bit of additional terminology. A ring R is called reduced if N = {0}, and R is called normal if all idempotents are central. An element *x* ∈ R is periodic if there exist distinct *m*, *n* ∈ $\mathbb{Z}^+$ for which $x^n - x^m$ ; and x is potent if there exists *n* ∈ $\mathbb{Z}^+$, *n* > 1, such that $x^n$ = *x*. It is easy to show that if R is reduced, every periodic element is potent. As usual, if x, y ∈ R, the symbol [*x*, *y*] represents the commutator *x y* – *y x*; and extended commutators $[x, y]_k$, *k* ≥ 1, are defined inductively by taking $[x, y]_1$ = [*x*, *y*] and $[x, y]_k$ = [$[x, y]_k$ −1, *y*]. It is easily verified that

$$[x, y]_k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\, y^i x\, y^{k-i}$$

The symbol C(R) denotes the commutator ideal of R, and $\langle x \rangle$ denotes the subring generated by *x*. Finally, the symbol ((*m*, *n*)) represents an ordered pair of positive integers of opposite parity.

We now state two lemmas which apply to rings which are not necessarily semiperiodic, followed by several lemmas dealing with elementary properties of semiperiodic rings.

If R is any ring in which each element is central or potent, then R is commutative.

If R is a ring containing an ideal I such that here both I and R/I are commutative, then the commutator ideal C(R) is nil and N is an ideal.

Let R be any semiperiodic ring.

**(i)** Every epimorphic image of R is semiperiodic, and every ideal of R is semiperiodic.
**(ii)** If e is any idempotent with additive order not a power of 2, then e ∈ Z.
**(iii)** If *x* ∉ J ∪ Z, there exists *q* ∈ $\mathbb{Z}^+$ and g(*t*) ∈ *t* $\mathbb{Z}$ [*t*] such that e = g(*x*) is idempotent and $x^q = x^q$e.

If R is a normal semiperiodic ring, then N ⊆ J.

If R is a normal semiperiodic ring and σ : R → S is a ring epimorphism, then N(S) ⊆ Z(S) ∪ σ(J).

If R is a semiperiodic ring with 1, then J ⊆ N or J ⊆ Z.

 

If R is a normal semiperiodic ring, then R/J is commutative. If in addition J is commutative, then N is an ideal and R/N is commutative.

If R is a reduced semiperiodic ring with R ≠J, then R is commutative.

If R is a 2-torsion-free semiprime semiperiodic ring with R ≠J, then R is commutative.

Let R be a semiperiodic ring with R ≠ J. If both R and R/J are 2-torsion-free, then R is commutative.

Let R be a semiperiodic ring with 1 which satisfies the following two conditions:

**(i)** For each $a \in$ N and $x \in$ R, there exists $k \in \mathbb{Z}^+$ for which $[a, x]_k = 0$;
**(ii)** N is commutative.

Then R is commutative.

Let R be a semiperiodic ring satisfying the following conditions:

**(i)** For each $a \in$ N and $x \in$ R, there exists $k \in \mathbb{Z}^+$)for which $[a, x]k = 0$;
**(ii)** J is commutative.

Then R is commutative.

**ON THE JACOBSON RADICAL AND UNIT GROUPS OF GROUP ALGEBRAS**

Let R be any associative ring with identity 1 ≠ 0. Then R may be treated as a Lie ring under the Lie multiplication $[x, y] = xy - yx$, $x, y \in$ R. The Lie ring thus obtained is denoted by L(R) and is called the associated Lie ring of R. The lower central chain $\{\gamma_n(L(R)) \mid n = 1, 2, \ldots\}$ and the derived chain $\{\delta^n(L(R)) \mid n = 1, 2, \ldots\}$ of L(R) are defined inductively as follows:

$\gamma_1(L(R)) = \delta^0(L(R)) = L(R)$ ,

$\gamma_{n+1}(L(R)) = [\gamma_n(L(R)), L(R)]$ ,

$\delta^n(L(R)) = [\delta^{n-1}(L(R)), \delta^{n-1}(L(R))]$ .

The Lie ring $L(R)$ is solvable of length $n$ if $\delta^n(L(R)) = (0)$ but $\delta^{n-1}(L(R)) \neq (0)$. Let J(R) denote the Jacobson radical of R. Then 1 + J(R) is a normal subgroup of the unit group U(R) and we have the exact sequence of groups

$$1 \to 1 + J(R) \to U(R) \to U(R) = J(R)) \to 1.$$

Thus $U(R)/(1 + J(R)) \cong U(R/J(R))$. If further 2 and 3 are invertible in R and the associated Lie ring L(R) is solvable, then $\gamma_2(L(R))R = \delta^1(L(R))R$ is a nil ideal of R. Since nil ideals are always contained in the Jacobson radical, we have, in this situation, $\gamma_2(L(R))R \subseteq J(R)$ and thus R/J(R) is commutative. Thus the commutator subgroup $U(R)'$ of U(R) is contained in 1+J(R). If J(R) is nilpotent as an ideal, then 1+J(R) is nilpotent as a group and so U(R) is solvable. In particular, in the above situation, if $(J(R))^2 = 0$, then U(R) is metabelian.

In this study some connections in the above direction when R = KG is the group algebra of the group G over the field K, where Char K = $p > 0$ and G is finite. Throughout, $Z_p$ denotes the field with $p$ elements.

Let KG be the group algebra of the group G over the field K. We denote by $\Delta(G)$, the augmentation ideal of KG. Clearly 1 + J(KG) defines a normal subgroup of the unit group U(KG). Also there are the trivial units of the form kg, 0 ≠ k ∈ K, g ∈ G, in U(KG). Our aim, in this chapter, is to investigate situations where U(KG) = K*G(1+J(KG)), K* = K\{0}. Obviously U(KG) cannot be smaller than this as the right hand side is always contained in U(KG).

 

Almost in all the known cases the Jacobson radical J(KG) of a group algebra KG is a nil ideal; and at least, for sure, this is the case for the class of solvable, linear and locally finite groups. Suppose Char K = *p*, *p* > 0 and J(KG) is nil. Then for any $\alpha \in$ J(KG), $\alpha^{p^n} = 0$ for some *n* $\geq$ 0 and thus $(1+\alpha)^{p^n} = 1+\alpha^{p^n} = 1$. This shows that 1+J(KG) is a normal p-subgroup of U(KG) if J(KG) is a nil ideal.

We make the following observations.

**Lemma 1:** Let K be a field with Char K = *p* > 0 and let G be agroup. Then G $\cap$ {1 + J (KG)} is a normal p-subgroup of G. Further if G is locally finite, then $O_p$(G) = G $\cap$ {1 + J(KG)}.

**Corollary 1:** If G is locally finite and Char K = *p* > 0, then $\Delta$(N)KG $\subseteq$ J(KG) for every normal *p*-subgroup N of G and equality holds if N is a normal Sylow *p*-subgroup of G.

It may be noted that $\Delta$(G) = J(KG) for any locally finite p-group G if Char K = *p* > 0.

**CONSEQUENCE**

Now start study of the problem: When U(KG) =K*G(1 +J(KG))

**Proposition1.1: Let** K is a field with Char K = *p* > 0 and let G be a locally finite group having a normal Sylow *p*-subgroup *P*. Then U(KG) =K*G(1 + J(KG)) if and only if one of the following holds:

**(i)** G = *P*;
**(ii)** K = $Z_2$ and G/*P* $\cong C_3$ ;
**(iii)** K = $Z_3$ and G/*P* $\cong C_2$.

**Theorem 1: If** Char K = *p* > 0 and G is a finite solvable group having no normal Sylow *p*-subgroup, then U(KG) = K*G(1+J(KG)) if and only if K = $Z_2$ and G=$O_2$(G) $\cong S_3$.

**Lemma 2: Let** G be a finite group and let Char K = *p* > 0 such that U(KG) = K*G(1 + J(KG)). Then U(K$\bar{G}$) = K* $\bar{G}$ (1 + J(K$\bar{G}$)), where $\bar{G}$= G/$O_p$ (G).

**Proof:**
Since

$$\Delta(O_p(G))KG \subseteq J(KG);$$

$$U(KG/J(KG)) \cong U(K\bar{G}/J(K\bar{G})).$$

Therefore,

$$\frac{U(KG)}{1+J(KG)} = \frac{K^*G(1+J(KG))}{1+J(KG)} \cong \frac{K^*G}{G\cap(1+J(Z_2G))}$$

$$= \frac{K^*G}{O_\square(G)} \cong \frac{U(K\bar{G})}{1+J(K\bar{G})}.$$

This clearly shows that U(K$\bar{G}$) =K*$\bar{G}$ (1 + J(K$\bar{G}$)).

When *p*'-elements are not central, A need not form a subgroup. Even when A forms a subgroup, Sylow *p*-subgroup need not be normal. However, we have the following.

**Theorem 2:** Let G be a finite group such that A forms a non-central subgroup and Char K = *P* > 0.If U(KG) = K*G(1 + J(KG)) then G is solvable and K is finite.

**We now discuss finite p-solvable groups:**

Let K be a field with Char K = *p* > 0 and G a finite group such that U(KG) is *p*-solvable. Then U( $Z_p$G) is *p*-solvable and hence U($Z_p$G/J($Z_p$G)) is *p*-solvable. But U($Z_p$G/J($Z_p$G))= $\prod_{i=1}^{r} GL_{n_i}(D_i)$, so each $D_i$ is a field, being a finite division ring. Thus for each *i*, $GL_{n_i}(D_i) = GL_{n_i}(GF(q_i))$, $q_i = p^{n_i}$ and *p*-solvabiblity forces each $n_i = 1$ or $n_i = 2$, $q_i = p$, $p = 2$ or 3. But $GL_2(Z_2)$ and $GL_2(Z_3)$ are solvable. Thus $U(Z_pG/J(Z_pG))$ is solvable and therefore, $U(Z_pG)$ is solvable. This gives that G is solvable. Thus U(KG) is p-solvable implies G is solvable. In particular, we have

**Theorem 3:** If Char K = $p > 0$ and G is a p-solvable group such that U(KG) =K*G(1 + J(KG)), then G is solvable.

**Proof:** Clearly U(KG) is *p*-solvable. Rest follows from the above discussion.

## CONDITIONS UNDER WHICH JACOBSON RADICAL OF THE GROUP ALGEBRA OF A GROUP IS EITHER COMMUTATIVE OR A CENTRAL SUB ALGEBRA

Over an algebraically closed field of characteristic it was not possible to typify completely those groups whose group algebras had commutative Jacobson radicals. On the other hand, over an algebraically closed field of odd characteristic the results for finite groups were complete, but for infinite groups it was not feasible to give conditions that were both necessary and sufficient; it was clear too that for infinite groups the prime 3 presented special difficulties. However it did appear that our necessary conditions were not far from being sufficient and in this study shall show that the appearances were not deceptive in the problem of centrality and, for odd primes, in the problem of commutativity. The earlier formulations of our results are not quite suitable for present purposes. According certain further conclusions, which were derived in the course of the original proofs, and incorporating some significant additions for which justification is subsequently given. We also take this opportunity to correct an error in previous.

This study considers a fixed algebraically closed field K of characteristic p > 0. The twisted group algebra of a group G over K is denoted by $K^t(G)$ , the ordinary group algebra being denoted simply by K(G), the corresponding Jacobson radicals are denoted by $JK^t(G)$ and JK (G) respectively, If A is a subgroup of G, |A| is the order of A and $C_G$ (A) is the centralizer of A in G. If B is also a subgroup of G, [A. B] is the subgroup generated by the elements $a^{-1}b^{-1}ab$ ($a \in A$, $b \in B$), in particular G' = [G, G] and G" = [G', G'] are the successive derived groups of G. P denotes a p Sylow subgroup of G and {e} is the trivial group. In the enunciations of the theorems and elsewhere it frequently happens that we describe the product RS of two finite subgroups R and S of G, R being normal in G, as being a Frobenius group with R as the Frobenius kernel. It is to be understood by this ellipsis that $R \cap S = \{e\}$, that $RS = R \cup \bigcup_{r \in R} r^{-1}Sr$ and that, for $x \in RS$, $S \cap \square^{-1}Sx \neq \{e\}$ if and only if $x \in S$; these conditions are equivalently expressed by saying that the inner automorphisms of RS induced by the elements of S act as regular automorphisms on R.

In order to avoid repetition this study assume henceforth, except in the last section, that G is a non-abelian group for which JK (G) is non-trivial.

**Theorem 4:** Let JK (G) be commutative. Then one of the following conclusions holds.

(1) P=2 and $[JK(G)]^2 = \{0\}$.

(2) |G'| is finite and divisible by p. p=2 or 3 and |P| = 2, 3 or 4. If p=3 and if G' is abelian then G'=P and G=$C_G$ (P). If p=3 and if G' is non-abelian then G'= G" P and G' is a Frobenius group with G" as the Frobenius kernel, further G= G" $C_G$ (P).

(3) | G'| is finite and not divisible by p. If≠p2 then P is finite and G' P is a Frobenius group in which G' is the Frobenius kernel.

**Theorem 5:** JK (G) is central if and only if G' is finite and one of the following statements holds.

**(1)** |G'| is divisible by p and p = 2 = |P|. If $G' \supset P$ then G' = G" P and G' is a Frobenius group with G" as the Frobenius kernel.

**(2)** |G'| is not divisible by p and |P| is finite. G' P is a Frobenius group with G' as the Frobenius kernel.

As compared with the earlier theorem we have extended in (1) the description of G' but no new argument is involved, the conclusion following from a previous discussion. We have also omitted an assertion that G/ G' be torsion-free, this assertion being dependent on a faulty argument . These modification to the original theorem do now however enable us to give group- theoretical conditions which are both necessary and sufficient. We are unable us give a strict converse to Theorem 6.1.1 owing to difficulty with the prime 2. This study do give below a partial converse and in the theorem that follows give necessary and sufficient conditions in the commutativity problem when $p \neq 2$, this theorem being itself an immediate consequence of Theorems 4 and 6.

**Theorem 6:** Any one of the following assumptions implies that JK (G) is commutative.

**(1)** p = 2 and $[JK(G)]^2 = \{0\}$.

**(2)** |G'| is finite and divisible by p. p = 2 or 3 and |P| = 2 or 3. If p = 3 and if G' is abelian then G' = P and G = $C_G$(P). If p = 3 and if G' is non-abelian then G' = G" P and G' is a Frobenius group with G" as the Frobenius kernel, further G = G" $C_G$(P).

(3) |G'| is finite and not divisible by p. |P| is finite and G' P is a Frobenius group in which G' is the Frobenius kernel.

**Theorem 7:** Let $p \neq 2$. Then JK(G) is commutative if and only if G' is finite and one of the following statements holds.

**(1)** |G'| is divisible by p and p = 3 = |P|. G' is abelian then G' = P and G = $C_G(P)$. If G' is non-abelian then G' = G" P and G' is a Frobenius group with G" as the Frobenius kernel, further G = G" $C_G$ (P).

**(2)** |G'| is not divisible by p and |P| is finite. GP is a Frobenius group in which G' is the Frobenius kernel.

If G' is non-non abelian in (1) of the theorem4 then G" admits a regular automorphism of order 3. Such a subgroup has been known to be nilpotent for some time. The nilpotency of G' in (2) of theorem4 is now also known. Consequently we obtain the next theorem whose proof directly from the above results.

**Theorem 8:** Either of the two following assumptions implies that G is a solvable FC-group.

**(1)** P is odd and JK(G) is commutative.
**(2)** JK(G) is central.

**PRELIMINARY LEMMAS**

In this section this study prove some result s that wiil be used later.

**Lemma 3:** Let H = MQ be a Frobenius group in which Q is a p-Sylow supbgroup of H and in which M is the Frobenius kernel. Then JK(H) = $[\sum_{x \in M} x]$JK(Q).

**Lemma 4:** Let M and Q be two finite subgroups of a group H, M being normal in H.

**(1)** If H = M$C_H$(Q) then [Q,H] $\subseteq$ M.

**(2)** If [Q,H] $\subseteq$ M and if MQ is a Frobenius group with M as Frobeneius kernel then H = M$C_H$(Q).

**Lemma 5:** Let M be a finite normal subgroup of a group H, |M| being prime to p, and let Q be a finite p-subgroup of H such that of H such That MQ is normal in H.

**(1)** Let $p \neq 2$ and let the ideal ($\sum_{x \in M} x$)JK(Q)K(H) be a commutative. Then [H,H] $\subseteq$ MQ and [Q,H] $\subseteq$ M.

**(2)** Let p = 3 = |Q|, let [H,H] $\subseteq$ MQ and [Q,H] $\subseteq$ M. Then the ideal ($\sum_{x \in M} x$) JK(Q)K(H) is commutative.

**Lemma 6:** Let p =3. Let Q be normal 3-subgroup of a group H and let |Q| = 3. Let JK (H) be a commutative. Then
(1) H = $C_H$(Q).
(2) [H, H] $\subseteq$ Q.

**Proof of Theorem 5:**
For convenience these studies prove this theorem first. From our previous remarks, the proof of theorem2 is completed if it is shown that both (1) and (2) of theorem2 imply that JK(G) is central . This study requires separating the two cases and we begin therefore by assuming that (1) holds. The abelian group G/G' has no non-trivial elements of order 2, for if

 

R/G' is a non-trivial 2-subgroup then R has order divisible by 4, contradicting the assumption that a 2-Sylow subgroup of G has order 2. Hence we have
JK(G) = JK(G') K(G). Since |P| = 2,

$$JK(P) = \{\lambda \sum_{x\in p} x : \lambda \in K\}.$$

Thus if G' = P then JK (G') = $\{\lambda \sum_{x\in G'} x : \lambda \in K\}$ and if G' $\supset$ P it follows, by lemma3 , that

$$\begin{aligned} JK(G') &= \left[\sum_{y\in G''} y\right] JK(P) \\ &= \left[\sum_{y\in G''} y\right]\left[\{\lambda \sum_{x\in P} x : \lambda \in K\}\right] \\ &= \{\lambda \sum_{z\in G'} z : \lambda \in K\}. \end{aligned}$$

Hence, whether G' equals P or not,

$$JK(G) = \left[\sum_{x\in G'} x\right] K(G), \text{ This is central ideal.}$$

Suppose now that (2) of theorem2 holds. Then G/( G' P) is an abelian group with no non-trivial p-elements and thus JK(G) = JK(G'P)K(G). Since G' P is Frobenius with G' as Frobenius kernel it follows, by lemma3, that

$$\begin{aligned} JK(G'P) &= \left[\sum_{x\in G'} x\right] JK(P) \text{ and hence} \\ JK(G) &= \left[\sum_{x\in G'} x\right] JK(P)\, K(G) \\ &\subseteq \left[\sum_{x\in G'} x\right] K(G), \text{ which is a central ideal.} \end{aligned}$$

We proceed to the proof of theorem4. by the remarks in section 1, theorem4 is proved once certain assertions concerning the prime 3 are proved. Thus suppose that JK(G) is commutative and that | G'| is finite and divisible by p where p = 3 = | P|. Since G' is finite and normal in G, JK(G') $\subseteq JK(G)$ and so JK(G') is commutative and non-trivial. If G' is abelian then P is normal in G and hence, by lemma6 G' = P and G = □□(P). If G' is non-abelian then G"P is a Frobenius group with G" as the Frobenius kernel. This implies, by lemma3, that JK(G"P) = $\left[\sum_{x\in G''} x\right]$ JK(P).

But G" P is a characteristic subgroup of G' and so G" P is a finite normal subgroup of G. Thus JK (G" P) $\subseteq$ JK (G) and so the ideal ( $\sum_{x\in G''} x$) JK(P)K(G) is commutative. By lemma5 (1) [G, G] $\subseteq$ G"P and [P,G] $\subseteq$ G". The first assertion implies that G' = G"P and the second assertion implies, by lemma4 (2), that G = G" $C_G$(P). This completes the proof of theorem4.

We now seek to establish theorem6. (1) of this theorem implies that JK(G) is trivially commutative. By theorem5, which has been proved, (3) again implies trivially that JK(G) is commutative. Assume therefore that (2) of theorem6 holds. Under this hypothesis G/ G' is abelian with no non-trivial elements of order p and hence JK(G) = JK(G')K(G). If p = 2 then $[JK(G')]^2$ = {0} and so $[JK(G)]^2$ = {0} rendering JK (G) commutative. Suppose now that p = 3. If $G'$ abelian then $G' = P$ and so JK(G) = JK(P)K(G).

By lemma5 (2), since G = $C_G$(P), JK(P)K(G) is commutative. If $G'$ is non abelian then, by lemma3 (1), JK($G'$) = JK(G" P) = $(\sum_{x\in G''} x) JK(P)$ and thus JK(G) = $(\sum_{x\in G''} x)\, JK(P)$ K(G). By lemma4 (1), since G = G"$C_G$(P), we have [P,G] $\subseteq$ G". As also [G,G] = G" P, we can apply lemma5 (2) to deduce that $(\sum_{x\in G''} x)\, JK(P)$ K(G) is commutative. This completes the proof.

In this section this study are not assuming that JK(G) is commutative or non-trivial. We address ourselves to the problem of determining conditions under which, if H is a finite normal subgroup of G, then JK(G) = JK(H) K(G), a necessary condition being of course that JK(G/H) ={0}. The issue seems to be difficult to settle for ordinary group algebras but if we consider all possible 'twists' on a group algebra and on certain subgroup algebras a solution is feasible.

## CONCLUSION

This study have covered most of the cases for finite groups except for finite groups which are not *p*-solvable, in which the *p*'-elements are non-central and do not form a subgroup. This study has considered a fixed algebraically closed field

K of characteristic $p > 0$. The twisted group of a group G over K is denoted by $K^t(G)$, the ordinary group algebra being denoted simply by K(G), the corresponding Jacobson radicals are denoted by J$K^t(G)$ and JK(G) respectively. Thus this study results that Jacobson radical of the Group Algebra of a Group is either commutative or a central sub algebra.

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