Research Journal of Pure Algebra -2(7), 2012, Page: 206-216
Available online through www.rjpa.info ISSN 2248–9037

Bounded Artex Spaces Over Bi-monoids and Artex Space Homomorphisms

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(Received on: 24-06-12; Accepted on: 02-07-12)

ABSTRACT

We introduce Some Special Artex Spaces over bi-monoids namely Complete Artex Spaces over bi-monoids , Lower Bounded Artex Spaces over bi-monoids, Upper Bounded Artex Spaces over bi-monoids, Bounded Artex Spaces over bi-monoids ,Artex Space homomorphism, Artex Space epimorphism, Artex Space monomorphism, and Artex Space isomorphism. We prove the homomorphic image of a Lower Bounded Artex Space over a bi-monoid is a Lower Bounded Artex Space over the bi-monoid and the homomorphic image of an Upper Bounded Artex Space over a bi-monoid is an Upper Bounded Artex Space over the bi-monoid and in general the homomorphic image of a Bounded Artex Space over a bi-monoid is a Bounded Artex Space over the bi-monoid. We prove under the Artex Space homomorphism the least element goes to the least element and the greatest element goes to the greatest element. Also we prove the cartesian product of Lower Bounded Artex Spaces over a bi-monoid is a Lower Bounded Artex Space over the bi-monoid and the Cartesian product of Upper Bounded Artex Spaces over a bi-monoid is an Upper Bounded Artex Space over the bi-monoid and in general the Cartesian product of Bounded Artex Spaces over a bi-monoid is a Bounded Artex Space over the bi-monoid.

1. INTRODUCTION

The study of Lattices and Boolean algebra is an interesting one for the algebraist. When George Boole introduced Boolean Algebra in 1854, it was new, but, nowadays, they have very important applications in the theory and design of computers. There are many other areas such as engineering and science to which Boolean algebra is applied. This motivated us to bring our previous paper titled "Artex Spaces over Bi-monoids", Research Journal of Pure Algebra, 2(5), May 2012, pages 135- 140. But a theory will help or will be useful or can lead other theories, if the theory itself is developed in its own way. As a development of it, now, we extend the theory of Artex spaces over bi-monoids further to Some Special Artex Spaces over bi-monoids. We hope the theory of Some Special Artex spaces over bi-monoids will play an important role in future and will be useful to computer fields. As special lattices namely complete lattice, bounded lattice and other lattices, our Special Artex Spaces over bi-monoids, in future, will play a good role in Discrete Mathematics, Science and Engineering, and in Computer fields.

2. PRELIMINARIES

2.1.0 Definitions and Examples

2.1.1 Definition: Doubly Closed Space: A non-empty set D together with two binary operations denoted by + and . is called a Doubly Closed Space if
(i) a.(b+c) = a.b + a.c and
(ii) (a+b).c = a.c + b.c, for all a, b, c ϵ D

A Doubly closed space is denoted by (D, + , .)

2.1.2 Example: (N, +, .), where N is the set of all natural numbers, is a Doubly closed space.

2.1.3 Definition: Bi-monoid: A system (M , + , .) is called a Bi-monoid if
1. (M, +) is a monoid
2. (M, .) is a monoid and
3. a.(b + c) = a.b + a.c and (a + b).c = a.c + b.c , for all a, b, c ϵ M.

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In other words, a Doubly closed space (M, +, .) is called a Bi-monoid if

1. (M, +) is a monoid and
2. (M, .) is a monoid

2.1.4 Example: Let W= {0, 1, 2, 3,…}.

Then (W, +, .), where + and **.** are the usual addition and multiplication respectively, is a bi-monoid.

2.1.5 Example: Let S be any set. Consider P(S), the power set of S.

Then P(S) is a bi-monoid under the operations union and intersection.

ie (P(S) ,∪, ∩) is a bi-monoid.

2.1.6 Example: 1. Let Q'= Q^+ ∪ {0}, where Q^+ is the set of all positive rational numbers.

Then Q' is a bi-monoid under the usual addition and multiplication.

ie (Q', + , .) is a bi-monoid.

2. Let R'= R^+ ∪ {0}, where R^+ is the set of all positive real numbers.

Then R' is a bi-monoid under the usual addition and multiplication.

ie (R', + , .) is a bi-monoid.

2.1.7 Definition: Lattice: A lattice is a partially ordered set (L, ≤) in which every pair of elements a, b ϵ L has a greatest lower bound and a least upper bound.

The greatest lower bound of a and b is denoted by aΛb and the least upper bound of a and b is denoted by avb

2.1.8 Definition: Lattice as Algebraic System: A lattice is an algebraic system (L , Λ , V) with two binary operations Λ and V on L which are both commutative, associative, and satisfy the absorption laws namely

aΛ (aVb) = a and aV(a Λb) = a

The operations Λ and V are called cap and cup respectively, or sometimes meet and join respectively.

2.1.9 Definition: Complete Lattice: A lattice is called a complete lattice if each of its nonempty subsets has a least upper bound and a greatest lower bound.

Every finite lattice is a complete lattice and every complete lattice must have a least element and a greatest element.

The least and the greatest elements, if they exist, are called the bounds or units of the lattice and are denoted by 0 and 1 respectively.

Note : The identity elements of the bi-monoid (M , + , .) with respect to + and . , if no confusion arises, are also denoted by 0 and 1 respectively.

2.1.10 Definition: Bounded Lattice: A lattice which has both elements 0 and 1 is called a bounded lattice.
A bounded lattice is denoted by (L, Λ, V, 0, 1)

The bounds 0 and 1 of a lattice (L, Λ, V) satisfy the following identities.

For any a ϵ L, a V 0 = a, a Λ 1 = a, a V 1 = 1, a Λ 0 = 0

2.1.11Definition: Artex Space Over a Bi-monoid: Let (M , + , .) be a bi-monoid. A non-empty set A is said to be an Artex Space Over the Bi-monoid (M, + , .) if

1. (A, Λ, V) is a lattice and

2. for each m ϵ M , m ≠ 0, and a ϵ A, there exists an element ma ϵ A satisfying the following conditions:

(i) m(a Λ b) = ma Λ mb
(ii) m(a V b) = ma V mb
(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a
(iv) (mn)a = m(na), for all m,n ϵ M, m≠0, n≠0, and a,b ϵ A
(v) 1.a = a , for all a ϵ A.

Here, ≤ is the partial order relation corresponding to the lattice (A, Λ , V)

The multiplication ma is called a bi-monoid multiplication with an artex element or simply bi-monoid multiplication in A.

Unless otherwise stated A remains as an Artex space with the partial ordering ≤ which need not be "less than or equal to" and M as a bi-monoid with the binary operations + and . need not be the usual addition and usual multiplication.

2.1.12 Example: Let W = {0, 1, 2, 3,…} and let Z be the set of all integers.

Then (W , + , .) is a bi-monoid , where + and . are the usual addition and multiplication respectively. (Z, ≤) is a lattice in which Λ and V are defined by a Λ b = mini {a, b} and a V b = maxi {a, b}, for all a, b ϵ Z.

Clearly for each m ϵ W, m ≠ 0, and for each a ϵ Z, there exists ma ϵ Z satisfying the following conditions:

(i) m(a Λ b) = ma Λ mb
(ii) m(a V b) = ma V mb
(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a
(iv) (mn)a = m(na), for all m,n ϵ W, m≠0, n≠0, and a, b ϵ Z
(v) 1.a = a , for all a ϵ Z

Therefore, Z is an Artex Space Over the Bi-monoid (W , + , .)

2.1.13 Example: As defined in Example 2.1.12, Q, the set of all rational numbers is an Artex space over the bi-monoid W

2.1.14 Example: As defined in Example 2.1.12, R, the set of all real numbers is an Artex space over the bi-monoid W

2.1.15 Example: As defined in Example 2.1.12, Q, the set of all rational numbers is an Artex space over the bi-monoid Q' = Q^+ ᴜ {0}

2.1.16 Example: As defined in Example 2.1.12, R, the set of all real numbers is an Artex space over the bi-monoid Q'= Q^+ ᴜ { 0 }

2.1.17 Example: As defined in Example 2.1.12, R, the set of all real numbers is an Artex space over the bi-monoid R' = R^+ ᴜ {0}

Proposition 2.2.1: Let (L, ≤) be a lattice in which Λ and V denote the operations of cap and cup respectively. For any a, b ϵ L, a ≤ b ‹=› a Λ b = a ‹=› a V b = b

3 SOME SPECIAL ARTEX SPACES OVER BI-MONOIDS

3.1 Complete Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Complete Artex Space if as a lattice, A is a complete lattice, that is each nonempty subset of A has a least upper bound and a greatest lower bound.

3.2 Remark: Every Complete Artex space must have a least element and a greatest element. The least and the greatest elements, if they exist, are called the bounds or units of the Artex space and are denoted by 0 and 1 respectively.

3.3 Lower Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Lower Bounded Artex Space over M if as a lattice, A has the least element 0.

3.3.1 Example: Let A be the set of all constant sequences (x_n) in [0, ∞) and let W = {0, 1, 2, 3,…}.

Define ≤ ', an order relation, on A by for (x_n), (y_n) in A, $(x_n) \leq' (y_n)$ means $x_n \leq y_n$, for each n

Where ≤ is the usual relation "less than or equal to "

Since the sequences in A are all constant sequences, $x_n \leq y_n$, for some n implies $x_n \leq y_n$, for each n

Therefore, $x_n \leq y_n$, for each n and $x_n \leq y_n$, for some n in this problem are the same.

Let x ϵ A, where x = (x_n)

Clearly $x_n \leq x_n$, for each n

So, $(x_n) \leq' (x_n)$

Therefore, ≤' is resflexive.

Let x, y ϵ A, where x = (x_n) and y = (y_n) be such that x ≤' y and y ≤' x , that is, $(x_n) \leq' (y_n)$ and $(y_n) \leq' (x_n)$.

Then $(x_n) \leq' (y_n)$ implies $x_n \leq y_n$, for each n

and $(y_n) \leq' (x_n)$ implies $y_n \leq x_n$, for each n

Now, $x_n \leq y_n$, for each n , and $y_n \leq x_n$, for each n ,implies $x_n = y_n$, for each n.

Therefore, $(x_n) = (y_n)$, that is x = y

Therefore, ≤' is anti-symmetric.

Let x, y, z ϵ A, where x = (x_n), y = (y_n) and z = (z_n) be such that x ≤'y and y ≤'z, that is, $(x_n) \leq' (y_n)$ and $(y_n) \leq' (z_n)$.

Then $(x_n) \leq' (y_n)$ implies $x_n \leq y_n$, for each n

$(y_n) \leq' (z_n)$ implies $y_n \leq z_n$, for each n

Now, $x_n \leq y_n$, for each n , and $y_n \leq z_n$, for each n, implies $x_n \leq z_n$ for each n.

Therefore, $(x_n) \leq' (z_n)$

Therefore, ≤' is transitive.

Hence, ≤' is a partial order relation on A

Now the cap ,cup operations are defined by the following:

(x_n) Λ (y_n) = (u_n), where u_n =mini { x_n , y_n }, for each n.

(x_n) V (y_n) = (v_n), where v_n =maxi { x_n , y_n }, for each n.

Clearly (A, ≤') is a lattice.

The bi-monoid multiplication in A is defined by the following :

For each m ϵ W, m ≠ 0, and x ϵ A, where x = (x_n), mx is defined by mx = m(x_n) = (mx_n).

Since (x_n) is a constant sequence belonging to A, (mx_n) is also a constant sequence belonging to A.

Therefore (mx_n) ϵ A

Let x, y ϵ A, where x = (x_n) , y = (y_n) and let m ϵ W,m≠0

Then, it is clear that

(i) m(x Λ y) = mx Λ my
(ii) m(x V y) = mx V my

(iii) $mx \wedge nx \leq (m+n)x$ and $mx \vee nx \leq (m+n)x$
(iv) $(mn)x = m(nx)$, for all m, n ϵ W, m ≠ 0, n ≠ 0, and x, y ϵ A
(v) $1.x = x$, for all x ϵ A

Therefore, A is an Artex space over W.

The sequence (0_n), where 0_n is 0 for all n, is a constant sequence belonging to A

Also $(0_n) \leq' (x_n)$, for all the sequences (x_n) belonging to in A

Therefore, (0_n) is the least element of A.

That is, the sequence 0, 0, 0 … is the least element of A

Hence A is a Lower Bounded Artex space over W.

3.3.2 Example: If A is the set of all constant sequences (x_n) in $[0,\infty)$ and if $Q' = Q^+ \cup \{ 0 \}$, then as defined in Example 3.3.1, A is a Lower Bounded Artex space over Q'.

3.3.3 Example: If A is the set of all constant sequences (x_n) in $[0,\infty)$ and if $R' = R^+ \cup \{ 0 \}$, then as defined in Example 3.3.1, A is a Lower Bounded Artex space over R'.

3.4 Upper Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be an Upper Bounded Artex Space over M if as a lattice, A has the greatest element 1.

3.4.1Example: Let A be the set of all constant sequences (x_n) in $(-\infty, 0]$ and let W = {0, 1, 2, 3,…}.

Define ≤', an order relation, on A by for (x_n), (y_n) in A, $(x_n) \leq' (y_n)$ means $x_n \leq y_n$, for n = 1,2,3,…

where ≤ is the usual relation "less than or equal to"

Then as in Example 3.3.1, A is an Artex space over W.

Now, the sequence (1_n), where 1_n is 0, for all n, is a constant sequence belonging to A

Also $(x_n) \leq' (1_n)$, for all the sequences (x_n) in A

Therefore, (1_n) is the greatest element of A.

That is, the sequence 0, 0, 0 … is the greatest element of A

Hence A is an Upper Bounded Artex Space over W.

3.4.2 Example: If A is the set of all constant sequences (x_n) in $(-\infty, 0]$ and if $Q' = Q^+ \cup \{0\}$, then as defined in

Example 3.4.1, A is an Upper Bounded Artex Space over Q'.

3.4.3 Example: If A is the set of all constant sequences (x_n) in $(-\infty, 0]$ and if $R' = R^+ \cup \{0\}$, then as defined in Example 3.4.1, A is an Upper Bounded Artex Space over R'.

3.5 Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Bounded Artex Space over M if A is both a Lower bounded Artex Space over M and an Upper bounded Artex Space over M.

3.5.1 Example:

Let A be the set of all constant sequences $([x_n])$ in $(Z_7, +_7)$, where $Z_7 = \{[0],[1],[2],[3],[4],[5],[6]\}$ and let

W = {0, 1, 2, 3,…}.

Define ≤', an order relation, on A by for $([x_n])$, $([y_n])$ in A, $([x_n]) \leq' ([y_n])$ means $x_n \leq y_n$, for each n

where ≤ is the usual relation "less than or equal to" As said in the Example 3.3.1, since the sequences in A are all constant sequences, $x_n \leq y_n$, for some n implies

$x_n \leq y_n$, for each n.

Let $x \in A$, where $x = ([x_n])$

Clearly $x_n \leq x_n$, for each n

So, $([x_n]) \leq' ([x_n])$

Therefore, $\leq'$ is reflexive.

Let $x, y \in A$, where $x = ([x_n])$ and $y = ([y_n])$ be such that $x \leq' y$ and $y \leq' x$, that is, $([x_n]) \leq' ([y_n])$ and $([y_n]) \leq' ([x_n])$.

Then $([x_n]) \leq' ([y_n])$ implies $x_n \leq y_n$, for each n

$([y_n]) \leq' ([x_n])$ implies $y_n \leq x_n$, for each n

Now, $x_n \leq y_n$, for each n, and $y_n \leq x_n$, for each n, implies $x_n = y_n$, for each n.

Therefore, $([x_n]) = ([y_n])$, that is $x = y$

Therefore, $\leq'$ is anti-symmetric.

Let $x, y, z \in A$, where $x = ([x_n])$, $y = ([y_n])$ and $z = ([z_n])$ be such that $x \leq' y$ and $y \leq' z$,

That is, $([x_n]) \leq' ([y_n])$ and $(y_n) \leq' (z_n)$.

Then $([x_n]) \leq' ([y_n])$ implies $x_n \leq y_n$, for each n

$([y_n]) \leq' ([z_n])$ implies $y_n \leq z_n$, for each n

Now, $x_n \leq y_n$, for each n, and $y_n \leq z_n$, for each n, implies $x_n \leq z_n$ for each n.

Therefore, $([x_n]) \leq' ([z_n])$

Therefore, $\leq'$ is transitive.

Hence, $\leq'$ is a partial order relation on A

Now the cap and cup operations on A are defined by the following:

$([x_n]) \wedge ([y_n]) = ([u_n])$, where $u_n = \text{mini}\{x_n, y_n\}$, for each n.

$([x_n]) \vee ([y_n]) = ([v_n])$, where $v_n = \text{maxi}\{x_n, y_n\}$, for each n.

Clearly $(A, \leq')$ is a lattice.

The bi-monoid multiplication in A is defined by the following:

For each $m \in W$, $m \neq 0$, and $x \in A$, where $x = ([x_n])$, mx is defined by $mx = m([x_n]) = ([mx_n])$.

Since $([x_n])$ is a constant sequence belonging to A, $([mx_n])$ is also a constant sequence belonging to A.

Therefore, $([mx_n]) \in A$, that is $mx \in A$.

Let $x, y \in A$, where $x = ([x_n])$, $y = ([y_n])$ and let $m \in W, m \neq 0$

Then, it is clear that

(i) $m(x \wedge y) = mx \wedge my$
(ii) $m(x \vee y) = mx \vee my$
(iii) $mx \wedge nx \leq (m+n)x$ and $mx \vee nx \leq (m+n)x$

(iv) (mn)x = m(nx) , for all m,n ϵ W, m≠0, n≠0, and x, y ϵ A
(v) 1.x = x, for all x ϵ A

Therefore, A is an Artex space over W.

The sequence ($[0_n]$), where 0_n is 0 for all n, is a constant sequence belonging to A

Also ($[0_n]$) ≤' ($[x_n]$) , for all the sequences ($[x_n]$) in A

Therefore, ($[0_n]$) is the least element of A

Therefore, A is a Lower Bounded Artex space over W.

The sequence ($[1_n]$), where 1_n is 6 for all n, is a constant sequence belonging to A

Also ($[x_n]$) ≤' ($[1_n]$), for all the sequences ($[x_n]$) in A

Therefore, ($[1_n]$) is the greatest element of A

Therefore, A is an Upper Bounded Artex Space over W.

Since A is both a Lower Bounded Artex Space over M and an Upper Bounded Artex Space over W, A is a Bounded Artex space over W.

3.6 Artex Space Homomorphism: Let A and B be two Artex spaces over a bi-monoid M, where Λ_1 and V_1 are the cap, cup of A and Λ_2 and V_2 are the cap, cup of B. A mapping f: A → B is said to be an Artex Space homomorphism if

1. f(a Λ_1b) = f(a) Λ_2f(b)
2. f(a V_1b) = f(a) V_2f(b)
3. f(ma)) = mf(a) , for all m ϵ M, m ≠ 0 and a,b ϵ A .

3.7 Artex Space Epimorphism: Let A and B be two Artex spaces over a bi-monoid M. An Artex Space homomorphism f: A → B is said to be an Artex Space epimorphism if the the mapping f: A → B is onto.

3.8 Artex Space Monomorphism: Let A and B be two Artex spaces over a bi-monoid M. An Artex Space homomorphism f: A → B is said to be an Artex Space monomorphism if the mapping f: A → B is one-one.

3.9 Artex Space Isomorphism: Let A and B be two Artex spaces over a bi-monoid M. An Artex Space homomorphism f: A → B is said to be an Artex Space Isomorphism if the mapping f: A → B is both one-one and onto, ie, f is bijective.

3.10 Isomorphic Artex Spaces: Two Artex spaces A and B over a bi-monoid M are said to be isomorphic if there exists an Artex Space isomorphism from A onto B or from B onto A.

Proposition 3.10.1: Let A be a Lower Bounded Artex space over a bi-monoid M and let B be an Artex space over M.

Let f: A → B be an Artex Space epimorphism of A onto B. Then B is a Lower Bounded Artex Space over M.

In other words, the homomorphic image of a Lower Bounded Artex Space over a bi-monoid is a Lower Bounded Artex space over the bi-monoid.

Proof: Let A be a Lower Bounded Artex Space over a bi-monoid M and let B be an Artex space over M.

Let f: A → B be an Artex Space epimorphism of A onto B.

Let $\leq_1$ and $\leq_2$ be the partial orderings of A and B respectively.

Let Λ_1 and V_1 be the cap and cup of A and let Λ_2 and V_2 be the cap and cup of B.

Claim: $x \leq_1 y$ => $f(x) \leq_2 f(y)$

By proposition 2.2.1, $x \leq_1 y$ <=> $x \Lambda_1 y = x$

Therefore, $f(x \wedge_1 y) = f(x)$

$f(x) \wedge_2 f(y) = f(x)$, (since f is a homomorphism)

Again by Proposition 2.2.1, $f(x) \leq_2 f(y)$

Let $b \in B$

Since f: A → B is an epimorphism of A onto B, there exists an element $a \in A$ such that f(a) = b

Since $0 \leq_1 a$, by the claim $f(0) \leq_2 f(a) = b$

That is, $f(0) \leq_2 b$, for all $b \in B$.

Since $0 \in A$, $f(0) \in B$

Therefore, f (0) is the least element of B

Hence B is a Lower Bounded Artex space over M.

Proposition 3.10.2: Let A be an Upper Bounded Artex space over a bi-monoid M and let B be an Artex space over M.

Let f: A → B be an Artex Space epimorphism of A onto B. Then B is an Upper Bounded Artex space over M.

In other words, the homomorphic image of an Upper Bounded Artex Space over a bi-monoid is an Upper Bounded Artex space over the bi-monoid.

Proof: Let A be an Upper Bounded Artex space over a bi-monoid M and let B be an Artex space over M.

Let f: A → B be an Artex Space epimorphism of A onto B.

Let $b \in B$

Since f: A → B is an epimorphism of A onto B, there exists an element $a \in A$ such that f(a) = b

Since $a \leq_1 1$, for all $a \in A$, $f(a) \leq_2 f(1)$

That is, $b \leq_2 f(1)$, for all $b \in B$.

Since $1 \in A$, $f(1) \in B$

Therefore, f (1) is the greatest element of B

Hence B is an Upper Bounded Artex space over M.

Proposition 3.10.3: Let A be a Bounded Artex space over a bi-monoid M and let B be an Artex space over M.

Let f: A → B be an Artex Space epimorphism of A onto B. Then B is a Bounded Artex space over M.

In other words, the homomorphic image of a Bounded Artex Space over a bi-monoid is a Bounded Artex space over the bi-monoid.

Proof: From the Propositions 3.10.1 and 3.10.2, it is clear that B is a bounded Artex space over M.

Proposition 3.10.4: Let A and B be Lower Bounded Artex spaces over a bi-monoid M. If f: A → B is an Artex Space epimorphism of A onto B, then f (0) = 0', where 0 and 0' are the least elements of A and B respectively.

Proof: Let A and B be Lower Bounded Artex spaces over a bi-monoid M

Let f: A → B be an Artex Space homomorphism.

Suppose 0 and 0' are the least elements of A and B respectively.

Since 0 is the least element of A, $0 \leq_1 a$, for all a ϵ A.

Since 0' is the least element of B, $0' \leq_2 b$, for all b ϵ B.

Since f is a homomorphism, $x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$,

Therefore, $0 \leq_1 a$, for all a ϵ A implies $f(0) \leq_1 f(a)$, for all a ϵ A.

Therefore, f (0) is the least element of f (A)

Since f: A → B is onto, f (A) = B.

Or in other way, if b ϵ B, since f is onto, there exists an element a ϵ A such that f(a) = b

But $0 \leq_1 a$, for all a ϵ A implies $f(0) \leq_1 f(a) = b$

That is, $f(0) \leq_1 b$, for all b ϵ B

Therefore, f (0) is the least element of B.

Hence f (0) = 0'.

Proposition 3.10.5: Let A and B be Upper Bounded Artex spaces over a bi-monoid M. If f: A → B is an Artex Space epimorphism of A onto B, then f (1) = 1', where 1 and 1' are the greatest elements of A and B respectively.

Proof: Let A and B be Upper Bounded Artex spaces over a bi-monoid M

Let f: A → B be an Artex homomorphism.

Suppose 1 and 1' are the greatest elements of A and B respectively.

Since 1 is the greatest element of A, $a \leq_1 1$, for all a ϵ A.

Since 1' is the greatest element of B, $b \leq_2 1'$, for all b ϵ B.

Since f is a homomorphism, $x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$, $a \leq_1 1$, for all a ϵ A implies $f(a) \leq_2 f(1)$, for all a ϵ A.

Therefore, f (1) is the greatest element of f(A)

Since f: A → B is onto, f(A) = B.

Therefore, f (1) is the greatest element of B.

Hence f (1) = 1'.

Proposition 3.10.6: If B and B' are any two Bounded Artex spaces over a bi-monoid M, then BXB' is also a Bounded Artex Space over M. If $\leq_1$ and $\leq_2$ are the partial orderings on B and B' respectively, then partial ordering ≤ on BXB' and the bi-monoid multiplication in BXB' are defined by the following:

For x, y ϵ BXB', where $x = (a_1,b_1)$ and $y = (a_2,b_2)$, $x \leq y$ means $a_1 \leq_1 a_2$, and $b_1 \leq_2 b_2$

For m ϵ M, m ≠ 0, and x ϵ BXB', where x= (a, b), the bi-monoid multiplication in BXB' is defined by

mx = m(a,b) = (ma,mb), where ma and mb are the bi-monoid multiplications in B and B' respectively.

In other words if Λ_1 and V_1 are the cap, cup of B and Λ_2 and V_2 are the cap, cup of B', then the cap, cup of BXB' denoted by Λ and V are defined by $x \Lambda y = (a_1,b_1) \Lambda (a_2,b_2) = (a_1 \Lambda_1 a_2 , b_1 \Lambda_2 b_2)$ and $x V y = (a_1,b_1) V (a_2,b_2) = (a_1 V_1 a_2 , b_1 V_2 b_2)$.

Proof: Let A = BXB'

We know that if (B, $\leq_1$) and (B', $\leq_2$) are any two Artex spaces over a bi-monoid M, then BXB' is an Artex space over M with the given partial ordering.

Therefore, it is enough to prove that A = BXB' is a Lower Bounded Artex Space over M and an Upper Bounded Artex Space over M.

(i) A = BXB' is a Lower Bounded Artex space over M.

Let x ϵ BXB', where x = (b, b')

Let $0_1$and 0_2 be the least elements of B and B' respectively

Then $0_1 \leq_1$ b, for all b ϵ B and $0_2 \leq_2$ b' , for all b' ϵ B'.

Let 0 = (0_1, 0_2).

Clearly 0 = (0_1, 0_2) ϵ BXB'.

Now, since $0_1 \leq_1$ b , for all b ϵ B and $0_2 \leq_2$ b' , for all b' ϵ B', (0_1, 0_2) $\leq$ (b,b') , for all (b, b') ϵ BXB'.

That is, 0 $\leq$ (b, b'), for all (b, b') ϵ BXB'.

That is, 0 $\leq$ x = (b, b'), for all x = (b, b') ϵ BXB'

Therefore, 0 is the least element of BXB'.

Therefore, A = BXB' is a Lower Bounded Artex space over M.

(ii) A = BXB' is an Upper Bounded Artex space over M.

Let 1_1 and 1_2 be the greatest elements of B and B' respectively.

Let 1 = (1_1, 1_2).

Clearly 1 = (1_1, 1_2) ϵ BXB'.

Now, since b $\leq_1 1_1$, for all b ϵ B and b' $\leq_2 1_2$, for all b' ϵ B', (b, b') $\leq$ (1_1, 1_2) , for all (b, b') ϵ BXB'.

That is, x = (b, b') $\leq$ 1 , for all x = (b, b') ϵ BXB'.

Therefore, 1 = (1_1, 1_2) is the greatest element of BXB'.

A = BXB' is an Upper Bounded Artex space over M.

Hence A = BXB' is a Bounded Artex space over M.

Corollary 3.10.7: If B_1, B_2, B_3… B_n are Bounded Artex spaces over a bi-monoid M, then B_1 X B_2 X B_3 X …..X B_n is also a Bounded Artex space over M.

Proof: The proof is by induction on n

When n=2, by the Proposition 3.10.6, B_1 X B_2 is a Bounded Artex space over M

Assume that B_1 X B_2 X B_3 X …..X B_{n-1} is a Bounded Artex space over M

Consider B_1 X B_2 X B_3 X …..X B_n

Let B= B_1 X B_2 X B_3 X …..X B_{n-1}

Then B_1 X B_2 X B_3 X …..X B_n = (B_1 X B_2 X B_3 X …..X B_{n-1}) X B_n = B X B_n

By assumption B is a Bounded Artex space over M

Again by the theorem B X B_n is a Bounded Artex space over M

Hence, B_1 X B_2 X B_3 X …..X B_n is a Bounded Artex space over M

Corollary 3.10.8: If B_1 and B_2 are Lower Bounded Artex spaces over a bi-monoid M, then B_1 X B_2 is also a Lower Bounded Artex space over M.

Proof: The proof is clear from the Proposition 3.10.6.

Corollary 3.10.9: If B_1, B_2, B_3… B_n are Lower Bounded Artex spaces over a bi-monoid M, then B_1 X B_2 X B_3 X …..X B_n is also a Lower Bounded Artex space over M.

Proof: The proof is clear from the Corollary 3.10.8 and the Corollary 3.10.7.

Corollary 3.10.10: If B_1 and B_2 are Upper Bounded Artex spaces over a bi-monoid M, then B_1 X B_2 is also an Upper Bounded Artex space over M.

Proof: The proof is clear from the Proposition 3.10.6.

Corollary 3.10.11: If B_1, B_2, B3... B_n are Upper Bounded Artex spaces over a bi-monoid M, then B_1 X B_2 X B_3 X …..X B_n is also an Upper Bounded Artex space over M.

Proof: The proof is clear from the Corollary 3.10.10 and the Corollary 3.10.7.

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Source of support: Nil, Conflict of interest: None Declared

