



COMMON FIXED POINT THEOREMS IN MENGER PROBABILISTIC QUASI METRIC SPACES UNDER WEAK COMPATIBILITY

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ABSTRACT

In this paper we prove a common fixed point theorem for weakly compatible maps in a Menger probabilistic quasi metric space. Incidentally, we observe that the result of Sunny Chauhan [12] is not valid through an example and modify it suitably.

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1. INTRODUCTION

Menger [7] introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [11]. The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points; but we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be of interest in the investigation of physical quantities and physiological threshold. It is also of fundamental importance in probabilistic functional analysis, non-linear analysis and applications [1, 2, 6].

In this section, some definitions and results in the theory of Menger probabilistic quasi metric spaces (briefly, Menger PQM-space) are given to fill in some background. For further information we refer to [3, 8, 10].

Definition 1.1 [11]: A mapping $T : [0,1] \times [0,1] \rightarrow [0,1]$ is called a triangle norm or t -norm if T satisfies the following conditions:

- (1) $T(a, 1) = a$ for all $a \in [0,1]$
- (2) $T(a, b) = T(b, a)$
- (3) $T(c, d) \geq T(a, b)$ for $c \geq a, d \geq b$
- (4) $T(T(a, b), c) = T(a, T(b, c))$ for all $a, b, c \in [0,1]$

The following are the four basic t -norms:

- (i) $T_M(x, y) = \text{Min}\{x, y\}$, (ii) $T_P(x, y) = x \cdot y$,
- (iii) $T(x, y) = \text{Max}\{x + y - 1, 0\}$

Each t -norm T can be extended [10] (by associativity) in a unique way to an n -ary operation taking for $(x_1, \dots, x_n) \in [0,1]^n$ ($n \in N$) the values

$$T^1(x_1, x_2) = T(x_1, x_2) \text{ and } T^n(x_1, \dots, x_{n+1}) = T(T^{n-1}(x_1, \dots, x_n), x_{n+1}) \text{ for } n \geq 2.$$

Definition 1.2: A t -norm T is of Hadzic-type (\mathcal{H} -type, in short) and $T \in \mathcal{H}$ if the family $\{T^n\}_{n \in N}$ of its iterates defined, for each x in $[0, 1]$, by $T^0(x) = 1, T^{n+1}(x) = T(T^n(x), x)$, for all $n \geq 0$, is equicontinuous at $x = 1$, that is given $\epsilon \in (0, 1), \exists \delta \in (0, 1)$ such that $x > 1 - \delta \implies T^n(x) > 1 - \epsilon$, for all $n \geq 1$.

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There is a nice characterization of continuous t -norm T of the class \mathcal{H} [9].

Definition1.3 [4]: If T is a t -norm and $(x_1, x_2, \dots, x_n) \in [0,1]^n$ ($n \in \mathbb{N}$), then $T_{i=1}^n x_i$ is defined recurrently by 1, if $n = 0$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 1$. If $(x_i)_{i \in \mathbb{N}}$ is a sequence of numbers from $[0, 1]$, then $T_{i=1}^\infty x_i$ is defined as $\lim_{n \rightarrow \infty} T_{i=1}^n x_i$ (this limit always exists) and $T_{i=n}^\infty x_i$ as $T_{i=1}^\infty x_{n+i}$.

Proposition 1.4: Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of numbers from $[0, 1]$ such that $\lim_{n \rightarrow \infty} x_n = 1$ and t -norm T be of \mathcal{H} -type. Then $\lim_{n \rightarrow \infty} T_{i=n}^\infty x_i = \lim_{n \rightarrow \infty} T_{i=1}^\infty x_{n+i} = 1$.

Definition 1.5: A mapping $F: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t \in \mathbb{R}} F(t) = 0$ and $\sup_{t \in \mathbb{R}} F(t) = 1$.

The Heaviside function H is a distribution function defined by $H(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$

The set of all distribution functions is denoted by \mathfrak{D} and $\mathfrak{D}^+ = \{F \in \mathfrak{D} / F(0) = 0\}$.

Definition 1.6 [8,10]: A Menger PQM-space is a triple (X, F, T) where X is a non empty set, T is a continuous t -norm and F is a mapping from $X \times X$ into \mathfrak{D}^+ such that, if $F_{p,q}$ denotes the value of F at (p, q) , then the following conditions hold:

(PQM1) $F_{p,q}(t) = F_{q,p}(t) = 1$ for all $t > 0$ iff $p = q$.

(PQM2) $F_{p,q}(t+s) \geq T(F_{p,r}(t), F_{r,q}(s))$ for all $p, q, r \in X$ and $t, s > 0$.

Definition 1.7 [8, 10]: Let (X, F, T) be a Menger PQM-space.

- (i) A sequence $\{x_n\}$ is said to be F -convergent to $x \in X$ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $k \in \mathbb{N}$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ whenever $n \geq k$.
- (ii) A sequence $\{x_n\}$ in X is called left Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $k \in \mathbb{N}$ such that $F_{x_r, x_s}(\varepsilon) > 1 - \lambda$ for all $s \geq r \geq k$.
- (iii) A Menger PQM-space (X, F, T) is called left complete if every left Cauchy sequence is F -convergent to a point in X .

In 1998, Jungck and Rhoades [5] introduced the following concept of weak compatibility.

Definition1.8 [5]: Let A and S be mappings from a Menger PQM-space (X, F, T) into itself. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is $Ax = Sx$ implies that $ASx = SAx$.

Sunny Chauhan [12] proved the following theorem.

Theorem 1.9: ([12], Theorem2.1) Let A, B, S, T and L be self maps of a complete Menger probabilistic quasi metric space (X, F, T) and suppose the following conditions are satisfied;

- a) $AB(X) \cup ST(X) \subseteq L(X)$
- b) $L(X)$ is a complete subspace of X
- c) The pairs (L, AB) and (L, ST) are weakly compatible
- d) $\min\{F_{ABx, STy}(t), F_{STx, ABx}(t)\} \geq 1 - \alpha(t)(1 - F_{Lx, Ly}(t))$

for all $x, y \in X$ and every $t > 0$, where $\alpha: \mathbb{R} \rightarrow (0,1)$ is a monotonic increasing function.

If $\lim_{n \rightarrow \infty} T_{i=n}^\infty (1 - \alpha^i(t)) = 1$, then A, B, S, T and L have a unique common fixed point.

The above theorem is not valid.

This is shown in the following example:

Example 1.10: If $X = [0, 1]$ and $Ax = 0, Bx = 1$ and $Lx = x$ for all x .

Also take $S = A$ and $T = B$. Then $ABx = A1 = 0 \forall x$

$BAx = B0 = 1 \forall x$. $F_{ABx, ABx}(t) = F_{0,0}(t) = 1$ and (AB, L) is weakly compatible.

Hence conditions (a), (b), (c) and (d) of Theorem 1.9 are satisfied for any relevant α and F. But A, B, S, T and L do not have a common fixed point.

2. MAIN RESULTS

We first prove a lemma.

Lemma 2.1: Suppose T is a continuous t- norm, $\{x_n\}$ is a sequence in $[0, 1]$ and $\lim_{n \rightarrow \infty} T_{i=n}^{\infty} x_i = 1$. Then $x_n \rightarrow 1$.

Proof: Let $\varepsilon > 0$. Then by hypothesis, there exists a positive integer N such that

$$T_{i=n}^{\infty} x_i > 1 - \varepsilon \text{ for all } n \geq N$$

so that $T_{i=n}^{n+k} x_i > 1 - \varepsilon$ for all $n \geq N$ and for every $k \geq 0$

Consequently, $x_n = T_{i=n}^n x_i > 1 - \varepsilon$ for every $n \geq N$.

Hence $x_n \rightarrow 1$ as $n \rightarrow \infty$.

Notation 2.2: Throughout the rest of the paper, T stands for a continuous t – norm which satisfies the condition:

$$\lim_{n \rightarrow \infty} T_{i=n}^{\infty} (1 - \alpha^i(t)) = 1 \text{ whenever } \alpha : (0, \infty) \rightarrow (0, 1) \dots\dots\dots(I)$$

From the above lemma, it follows that $\alpha^n(t) \rightarrow 1$ as $n \rightarrow \infty$ if T satisfies the above condition (I).

Lemma 2.3: Let (X, F, T) be a Menger PQM-space. If a sequence $\{x_n\}$ in X is such that for every $n \in \mathbb{N}$

$$F_{x_n, x_{n+1}}(t) \geq 1 - \alpha^n(t)(1 - F_{x_0, x_1}(t)) \text{ for every } t > 0,$$

where $\alpha : (0, \infty) \rightarrow (0, 1)$ is a monotonic increasing function then the sequence $\{x_n\}$ is a left Cauchy sequence.

Proof: For every $m > n$, write $m = n + k$. Let $\varepsilon > 0$ and $t > 0$. Then

$$\begin{aligned} F_{x_n, x_{n+k}}(t) &\geq T_{i=0}^{k-1} (F_{x_{n+i}, x_{n+i+1}}(\frac{t}{k})) \\ &\geq T_{i=0}^{k-1} (1 - \alpha^{n+i}(\frac{t}{k})(1 - F_{x_0, x_1}(\frac{t}{k}))) \\ &\geq T_{i=0}^{k-1} (1 - \alpha^{n+i}(\frac{t}{k})) \\ &\geq T_{j=n}^{n+k-1} (1 - \alpha^j(\frac{t}{k})) \text{ (put } n + i = j) \\ &= T_{j=n}^{m-1} (1 - \alpha^j(\frac{t}{m-n})) \\ &\geq T_{j=n}^{m-1} (1 - \alpha^j(t)) \text{ } (\because \alpha \text{ is monotonic increasing}) \\ &\geq T_{j=n}^{\infty} (1 - \alpha^j(t)) \\ &> 1 - \varepsilon \text{ (for } n \geq N, \text{ since (I) holds)} \end{aligned}$$

Hence the sequence $\{x_n\}$ is a left Cauchy sequence.

Corollary 2.4: Let (X, F, T) be a Menger PQM-space. If the sequence $\{x_n\}$ in X is such that for every $n \in \mathbb{N}$

$$F_{x_n, x_{n+1}}(t) \geq 1 - \alpha^n(t) \text{ for every } t > 0,$$

where $\alpha : (0, \infty) \rightarrow (0, 1)$ is a monotonic increasing function, then the sequence $\{x_n\}$ is a left Cauchy sequence.

Proposition 2.5: If (X, F, T) is a PQM-space then (X, E, T) is a Menger probabilistic metric space where $E_{x,y}(t) = \min\{F_{x,y}(t), F_{y,x}(t)\}$

Proof: $E_{x,y}(t) = \min\{F_{x,y}(t), F_{y,x}(t)\}$ for $x, y \in X$

Clearly $E_{x,y}(t) = E_{y,x}(t)$

$$\begin{aligned} E_{x,y}(t) = 1 &\Leftrightarrow F_{x,y}(t) = 1 \text{ and } F_{y,x}(t) = 1 \\ &\Leftrightarrow x = y \end{aligned}$$

$$\min\{F_{x,z}(t+s), F_{z,x}(t+s)\} \geq T(E_{x,y}(t), E_{y,z}(s)) \\ = T(\min\{F_{x,y}(t), F_{y,x}(t)\}, \min\{F_{y,z}(s), F_{z,y}(s)\})$$

$$\text{and } F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s)) \geq T(E_{x,y}(t), E_{y,z}(s))$$

$$F_{z,x}(s+t) \geq T(F_{z,y}(t), F_{y,x}(t)) \geq T(E_{z,y}(s), E_{y,x}(t))$$

$$\therefore E_{x,z}(t+s) \geq T(E_{x,y}(t), E_{y,z}(s))$$

\therefore If (X, F, T) is a PQM-space then (X, E, T) is a Menger probabilistic metric space.

Definition 2.6: E is called the induced probabilistic metric on X induced by the quasi probabilistic metric F.

Theorem 2.7: Let (X, F, T) be a complete Menger probabilistic metric space and let $g, L: X \rightarrow X$ be maps that satisfy the following conditions:

- (a) $g(X) \subseteq L(X)$;
- (b) $L(X)$ is a complete subspace of X ;
- (c) $F_{gx,gy}(t) \geq 1 - \alpha(t)(1 - F_{Lx,Ly}(t))$ for each $t > 0$ and for all $x, y \in X$, where $\alpha: (0, \infty) \rightarrow (0,1)$ is a monotonic increasing function.

If $\lim_{n \rightarrow \infty} T_{i=n}^{\infty} (1 - \alpha^i(t)) = 1$ then g and L have a unique point of coincidence. If further

- (d) The pair (L, g) is weakly compatible, then L and g have a unique common fixed point.

Proof: Let $x_0 \in X$. From condition (a), we can find x_1 , such that $L(x_1) = g(x_0)$.

Inductively, we define a sequence $\{x_n\}$ such that $Lx_{n+1} = gx_n$ for $n = 0, 1, 2, \dots$

Now, by taking $x = x_{n-1}$ and $y = x_n$ in (c), we get, for $t > 0$,

$$F_{gx_{n-1},gx_n}(t) \geq 1 - \alpha(t)(1 - F_{Lx_{n-1},Lx_n}(t)) \\ \geq 1 - \alpha(t) \left(1 - \left(1 - \alpha(t)(1 - F_{Lx_{n-2},Lx_{n-1}}(t)) \right) \right) \\ \geq 1 - \alpha^2(t)(1 - F_{Lx_{n-2},Lx_{n-1}}(t))$$

and by induction we get

$$F_{Lx_n,Lx_{n+1}}(t) \geq 1 - \alpha^n(t)(1 - F_{Lx_0,Lx_1}) \text{ for } n = 1, 2, \dots$$

Now by Lemma 2.3, $\{Lx_n\}$ is a left Cauchy sequence.

Since the space $L(X)$ is left complete,

there exists $z \in L(X)$, such that $\lim_{n \rightarrow \infty} Lx_n = z$.

Hence $\lim_{n \rightarrow \infty} gx_{n-1} = \lim_{n \rightarrow \infty} Lx_n = z$.

Since $z \in L(X)$, it follows that there exists $v \in X$ such that $L(v) = z$.

We prove that $gv = z$.

Put $x = x_{n-1}$ and $y = v$ in (c). We get

$$F_{gx_{n-1},gv}(t) \geq 1 - \alpha(t)(1 - F_{Lx_n,Lv}(t))$$

On letting $n \rightarrow \infty$ we get

$$F_{z,gv}(t) \geq 1 - \alpha(t)(1 - F_{z,z}(t))$$

so that $F_{z,gv}(t) = 1$. This is true for all $t > 0$.

$$\therefore gv = z.$$

Thus v is a point of coincidence to g and L . Suppose $gx = Lx$.

$$\begin{aligned} \text{Then, } F_{gx,gv}(t) &\geq 1 - \alpha(t)(1 - F_{Lx,Lv}(t)) \\ &= 1 - \alpha(t)(1 - F_{gx,gv}(t)) \end{aligned}$$

$$\text{So that } 0 \geq (1 - \alpha(t)) (1 - F_{gx,gv}(t))$$

$$\text{Consequently } 1 - F_{gx,gv}(t) = 0 \quad \forall t > 0.$$

$$\text{Similarly, we can show that } 1 - F_{gv,gx}(t) = 0 \quad \forall t > 0.$$

Hence $gx = gv$. Thus g and L have a unique coincidence point.

Suppose the pair (L, g) is weakly compatible.

Then $gv = z = Lv$ implies that $Lg(v) = gL(v)$ so that $Lz = gz$.

Thus, z and gz are coincidence points of g and L .

Hence $gz = z$ so that $Lz = gz = z$.

Consequently z is a common fixed point of L and g .

Since, under (d) every fixed point is a coincidence point, L and g have unique common fixed point.

Now we have the following corollary which can be treated as a modification of Theorem 1.9.

Corollary 2.8: Let A, B, S, T and L be self maps of a complete Menger probabilistic quasi metric space (X, F, T) and suppose the following conditions are satisfied:

- (i) $AB(X) \cup ST(X) \subseteq L(X)$;
- (ii) $L(X)$ is a complete subspace of X ;
- (iii) The pairs (L, AB) and (L, ST) are weakly compatible;
- (iv) $\min\{F_{ABx,STy}(t), F_{STx,ABy}(t)\} \geq 1 - \alpha(t)(1 - F_{Lx,Ly}(t))$ for all $x, y \in X$ and every $t > 0$, where $\alpha: (0, \infty) \rightarrow (0, 1)$ is a monotonic increasing function.
- (v) A, B and L commute and S, T and L commute.

If $\lim_{n \rightarrow \infty} T_{i=n}^{\infty} (1 - \alpha^i(t)) = 1$, then A, B, S, T and L have a unique common fixed point.

Proof: If we put $x = y$ in condition (iv) of the above theorem we obtain that

$$\min\{F_{ABy,STy}(t), F_{STy,ABy}(t)\} \geq 1 - \alpha(t)(1 - F_{Ly,Ly}(t))$$

$$\text{so that } F_{ABy,STy}(t) = 1 \text{ and } F_{STy,ABy}(t) = 1$$

$$\text{Hence } AB y = ST y \text{ for all } y \in X.$$

$$\text{Thus } AB = ST.$$

If we take, $AB = g$ in Theorem 2.7, AB and g have a unique common fixed point, say z .

$$\text{Then } AB z = L z = z$$

$$\text{So that } AB(Az) = AABz = ALz = Az = ALz = L(Az).$$

Thus Az is also fixed point of AB and L .

Similarly Bz is also fixed point of AB and L .

By the uniqueness of fixed point, follows that $Az = z = Bz = Lz$.

Thus z is the unique common fixed point of A , B and L .

Similarly we can show that z is the unique common fixed point of S , T and L . (since $AB = ST$).

Thus A , B , S , T and L have unique common fixed point.

NOTE:

(i) It may be observed that in Example 1.10, A and B do not commute.

(ii) Even though $AB = ST$ in corollary 2.8, it can be shown that A , B , S , T may be different, in view of the following Example.

Example 2.9: Let $X = [0, 1]$, $Ax = 0$, $Bx = x^3$, $Sx = 0$, $Tx = x^2$ and $Lx = x$ for all $x \in X$. Then $AB = ST$ on X and all the hypotheses of corollary 2.8 is satisfied. But $B \neq T$ so that the triad (A, B, L) is not equal to the triad (S, T, L) .

However A , B , S , T and L have unique common fixed point, namely, 0 .

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