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A COMMON FIXED POINT THEOREM FOR TWO SELF MAPS ON A CONE METRIC SPACE WITH CONVEX PROPERTY

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ABSTRACT

In this paper we introduce the notion of convex property in cone metric spaces. We show that a cone convex metric space has convex property. We also show that there are cone metric spaces with convex property, which are not cone convex metric spaces. We prove a common fixed point theorem for two self maps on a cone metric space with convex property and obtain the corresponding result on cone convex metric spaces as a corollary. An open problem is also given at the end of the paper.

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Key words: *Convex property, convex structure, Cone convex metric space, common fixed point theorem.*

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1. INTRODUCTION

In recent years, several authors (see [1-8]) have studied the strong convergence to a fixed point in cone metric spaces. Seong Hoon Cho and Mi Sun Kim [8] have proved certain fixed point theorems by using multi valued mappings in metric spaces and Ismat Beg and Akbar Azam [2] have proved certain fixed point theorems by using Kannan mapping in Convex metric spaces. R. Krishna Kumar and M. Marudai [4] introduced the notion of convex structure in cone metric spaces and proved fixed point theorems in cone convex metric spaces (that is, cone metric spaces with convex structure). In this paper we introduce (i) the notion of convex property in cone metric spaces (ii) show that every cone convex metric space is a cone metric space with convex property (iii) provide an example of a cone metric space with convex property which is not a cone convex metric space and (iv) prove a fixed point theorem for two self maps on a cone metric space with convex property. We observe that the fixed point theorem of R. Krishna Kumar and M. Marudai [4] is a special case of our result. We conclude the paper with an open problem.

We first recall definitions and known results that are needed in the sequel.

2. PRELIMINARIES

Huang Gaung, Zhang Xian [1] introduced the notion of a cone metric space.

Let E be a real Banach space. A subset P of E is said to be a cone if it satisfies the following conditions,

1) $P \neq \emptyset$ and P is closed.
2) $ax + by \in P$ for all $x, y \in P$ and for all non negative real numbers a and b.
3) $P \cap (-P) = \{0\}$.

Define partial ordering $\leq$ in E with respect to the cone P by $x \leq y$ if and only if $y - x \in P$. If $y - x \in$ interior of P then we write $x \ll y$. The cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$, implies $\|x\| \leq K\|y\|$.

The cone P is called regular if every increasing sequence which is bounded above is convergent. We observe that in a regular cone every decreasing sequence which is bounded below is convergent.

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2.1 Definition: [1] Let X be a non empty set. A mapping $d: X \times X \to E$ is said to be a cone metric if it satisfies,

(i) $0 \leq d(x,y) \ \forall \ x,y \in X \ and \ d(x,y) = 0 \ iff \ x = y$

(ii) $d(x,y) = d(y,x) \ \forall \, x,y \in X$

(iii) $d(x,y) \leq d(x,z) + d(z,y) \forall \, x,y,z \in X$, (X,d) is called a cone metric space.

2.2 Example: [1] Let $E = \mathbb{R}^2, P = \{(x,y) \in E : \ x,y \geq 0\}$, $X = \mathbb{R}^2$ and $d: X \times X \to E$ be defined by $d(x,y) = (|x-y|, \alpha|x-y|)$ when $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space. It is known that

2.3 Every regular cone is normal [1,7].

2.4 There is no normal cone with normal constant $M < 1$ [7].

2.5 For each $k > 1$, there is a normal cone with normal constant $M > k$ [7].

2.6 Example: [7] Let E = C_R ([0,1]) = { f : [0,1] → $\mathbb{R}$, f is continuous } with the supremum norm and $P = \{ f \in E : f \geq 0 \}$.

Then, P is a cone with normal constant of $M = 1$. Now define the sequence $\{f_n\}$ of elements of E by

$f_n(t) = t^n \ \forall \, t \in [0,1]$.

Then $\{f_n\}$ is decreasing and bounded from below but it is not convergent in E.

Hence this cone is not regular.

Thus a normal cone need not be regular.

The following is an example of a cone which is not normal.

2.7 Example: [7] Let $E = C_R^{\,'}$ $([0,1]) = \{f \, / f: \mathbb{R} \to [0,1]$, is continuously differentiable with the norm

$\|f\| = \|f\|_\infty + \|f'\|_\infty$

where $\|f\|_\infty = sup\{ |f(t)| : t \in [0,1] \}$ and consider the cone

$P = \{f \in E : f \geq 0 \}$.

For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then, $0 \leq g \leq f$,

$\|f\| = 2$ and $\|g\| = 2k + 1$. Since $k\|f\| < \|g\|$, k is not a normal constant of P.

Therefore, P is a non normal cone.

2.8 Definition: [1] Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then,

(i) $\{x_n\}$ converges to x when ever for every $c \in E$ with $0 << c$ there is a natural number N such that

$$d(x_n, x) << c \text{ for all } \geq N .$$

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x_m) << c \ \forall \ n,m \geq N$.

(iii) (X,d) is a complete cone metric space, if every Cauchy sequence in (X,d) is convergent in X.

R. Krishna Kumar and M. Marudai [4] introduced the notion of convex structure in a cone metric space.

2.9 Definition: [4] Let (X,d) be a cone metric space and $I = [0,1]$. A continuous mapping $R: X \times X \times I \to X$ is said to be a convex structure on X if for each $(x,y,\lambda) \in X \times X \times I$ and $u \in X$, $d(u, R(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)\, d(u,y)$.

A cone metric space X together with a convex structure R is called a cone convex metric space.

A nonempty subset L of X is said to be convex if $R(x,y,\lambda) \in L$ for all $(x,y,\lambda) \in L \times L \times I$.

2.10 Definition: We say that a sequence $\{\alpha_n\}$ of non negative real numbers is bounded away from zero, if the sequence $\{\alpha_n\}$ is having lower bound which is strictly greater than zero.

R. Krishna Kumar and M. Marudai [4] proved the following fixed theorem for a pair of self maps on a cone convex metric space.

2.11 Theorem: [4] Let A be a nonempty closed subset of a complete cone convex metric space X with convex structure R and let $S,T : X \to X$ be self mappings satisfying,

$$d(Sx,Ty) \leq \gamma[\, d(x,y) + d(x,Ty) + d(y,Sx)\,]$$

for all $x,y \in A$ and for some $0 < \gamma < 1$.

Suppose $x_0 \in X$. Define $\{x_n\}$ by,

$$x_{n+1} = R(Ty_n\,, x_n\,, \alpha_n);\; n = 0,1,2, \dots..$$

$$y_n = R(Sx_n\,, x_n\,\,, \beta_n); n = 0,1,2, \dots ....$$

where $0 \leq \alpha_n\,, \beta_n \leq 1$ and $\{\alpha_n\}$ is bounded away from zero.

If $\{x_n\}$ converges to some point $p \in A$ then p is a common fixed point of S and T .

3. MAIN RESULT

In this section we introduce the notion of convex property *(C.P)* in a cone metric space and show that convex property is more general than convex structure. We prove a fixed point theorem and obtain the result of R. Krishna Kumar et.al.[4] as a corollary. (Theorem 2.11)

Now we introduce the notion of convex property in cone metric spaces.

3.1 Definition: Let (X,d) be a cone metric space. Suppose to each $x,y \in X$ and $\lambda \in [0,1], \exists\, z \in X$ such that

$d(x,z) = \lambda\, d(x,y)$ and $d(y,z) = (1-\lambda)d(x,y)$

Then we say that (X,d) has convex property *(C.P)*.

We denote such a *z* by $C(x,y,\lambda)$ and call it a convex point associated with x,y,λ.

We observe that such a convex point z need not be unique.

3.2 Example: Let *X* be the unit circle in $\mathbb{R}^2$

For $x,y \in X$, define $d(x,y) =$ the length of the shorter arc, joining *x* and *y*.

Then (X,d) is a cone metric space with $E = \mathbb{R}$

Let $\lambda \in [0,1]$. Then there exists $z \in X$ such that

$$d(x,z) = \lambda\, d(x,y) and\; d(z,y) = (1-\lambda)d(x,y).$$

Thus (X,d) has convex property.

If x,y are diametrically opposite points on *X*, then clearly there exist $z\,, z\,', z \neq z\,'$, such that

$$d(x,z) = \lambda\, d(x,y)\,, d(z,y) = (1-\lambda)\, d(x,y).$$

$$d(x,z\,') = \lambda\, d(x,y)\,, d(z\,',y) = (1-\lambda)\, d(x,y).$$

Thus for given(x, y, λ), the point $C(x, y, \lambda)$ need not be unique.

Now we show that (X, d) of example 3.2 is not a cone convex metric space.

Suppose X is cone convex metric space. Then $\exists\ R: X \times X \times [0,1] \rightarrow X$ such that

$$d(u, R(x, y, \lambda)) \leq \lambda\, d(u, x) + (1-\lambda)\, d(y, u)\ \forall\ u \in X \tag{3.2.1}$$

Put $z = R(x, y, \lambda)$

This implies that $d(x, z) = (1-\lambda)\, d(x, y)$

$$d(z, y) = \lambda\, d(x, y),$$

In particular if x and y are diametrically opposite, $\lambda = \frac{1}{2}$ and u is diametrically opposite to z, we get from (3.2.1)

$\pi \leq \frac{1}{2}\frac{\pi}{2} + \frac{1}{2}\frac{\pi}{2}$ (since X is the unit circle in $\mathbb{R}^2$)

$=> \pi \leq \frac{2\pi}{4}$

$=> \pi \leq \frac{\pi}{2}$, a contradiction.

Therefore X is not a cone convex metric space.

Now we show that every cone convex metric space is a cone metric space with convex property.

3.3 Theorem: Every cone convex metric space is a cone metric space with convex property.

Proof: Let(X, d) be a cone convex metric space. Then there exists a function $R = R(x, y, \lambda) : X \times X \times [0,1] \rightarrow X$ such that,

$$d(u, R(x, y, \lambda)) \leq \lambda\, d(u, x) + (1-\lambda)\, d(u, y)\ \forall\ u \in X \tag{3.3.1}$$

On taking $u = x$ and $u = y$ respectively we get,

$$d(x, R(x, y, \lambda)) \leq (1-\lambda)\, d(x, y) \tag{3.3.2}$$

$$d(y, R(x, y, \lambda)) \leq \lambda\, d(x, y) \tag{3.3.3}$$

and now by triangle inequality,

$$\begin{aligned} d(x, y) &\leq d(x, R(x, y, \lambda)) + d(R(x, y, \lambda), y) \\ &\leq (1-\lambda)\, d(x\,, y) + \lambda\, d(x\,, y) \\ &= d(x\,, y) \end{aligned}$$

$$=> \ d(x, y) = d(x, z) + d(z, y) \tag{3.3.4}$$

where $z = R(x, y, \lambda)$

Claim: $d(x, z) = (1-\lambda)\, d(x, y)$ and $d(z\,, y) = \lambda\, d(x\,, y)$
Now, $d(x\,, z) \neq (1-\lambda)\, d(x\,, y)$

$=> d(x\,, z) < (1-\lambda)\, d(x\,, y)$ (by (3.3.2))

$=> \ d(x\,, z) + d(z\,, y) < (1-\lambda)\, d(x\,, y) + \lambda\, d(x\,, y)$ (by (3.3.3))

$=> \ d(x\,, y) < d(x\,, y)$ (by (3.3.4))
which is a contradiction.

Therefore $d(x\,,z) = (1-\lambda)\, d(x\,,y)$

Similarly $d(z,y) = \lambda\, d(x,y)$

Thus (X,d) is a cone metric space with convex property.

Note: Example 3.2 and theorem 3.3 show that the notion of convex property is a proper generalization of the notion of convex structure.

Now we prove a fixed point theorem for two self maps on a cone metric space with convex property.

3.4 Theorem: Let (X,d) be a cone metric space with property $(C.P)$ and $C(x,y,\lambda)$ be an associated convex point with (x,y,λ).

Let $S,T : X \to X$ be self maps such that for some $k \in (0,1)$

$d(Sx,Ty) \leq k\,[\,d(x,y) + d(x,Ty) + d(y,Sx)\,]\; for\ all\ x,y \in X.$

Suppose that $\{x_n\}$, associated with *S* and *T*, is defined by

(1) $x_0 \in X$

(2) $y_n = C(Sx_n\,,x_n\,,\beta_n)$ where $n = 0,1,2,\ldots.$

(3) $x_{n+1} = C(x_n\,,Ty_n,\alpha_n)\; where\ n = 0,1,2,\ldots.$

where $0 < \alpha_n\,,\beta_n < 1$ and $\{\,\alpha_n\}$ is bounded away from zero.

If $\{x_n\}$ converges to some point $p \in X$ then p is a common fixed point of *S* and *T*.

Proof: Suppose $x_n \to p$.

Let $0 << c$ be given.

Since $\{x_n\}$ is bounded away from zero, $\exists\ \lambda \ni 0 < \lambda \leq \alpha_n$ for every *n*. Choose a natural number *N* such that

$d(x_n\,,p) << \dfrac{\lambda c}{2}\ \forall\, n \geq N.$

$d(x_n,x_{n+1}) = d(x_n, C(x_n,Ty_n,\alpha_n))$

$= \alpha_n\, d(x_n,Ty_n) \geq \lambda d(x_n,Ty_n)$

so that $d(Ty_n,x_n) \leq \frac{1}{\lambda}\, d(x_n,x_{n+1})$.

Now $d(x_n\,,x_{n+1}) \leq d(x_n\,,p) + d(p,x_{n+1})\; for\ \ n \geq N$

$<< \frac{\lambda c}{2} + \frac{\lambda c}{2} = \lambda c$

Therefore $d(Ty_n,x_n) \leq \frac{1}{\lambda}\, d(x_n\,,x_{n+1})$

$<< \frac{1}{\lambda}\,\lambda c = c\ \ \forall\ \ n \geq N$

Hence $Ty_n \to p$

Therefore by (2) and (3)

$d(x_n,y_n) = d(\,x_n\,,C(Sx_n\,,x_n\,,\beta_n))$

$= (1-\beta_n)\, d(\,x_n\,,Sx_n)$

$d(Sx_n\,, y_n) = d(Sx_n\,, C(Sx_n\,, x_n\,, \beta_n))$

$\qquad = \beta_n\, d(\,x_n\,, Sx_n)$

Therefore $d(x_n\,, y_n) + d(Sx_n\,, y_n) = d(x_n\,, Sx_n)$

$d(Sx_n\,, Ty_n) \le k\,[\,d(x_n\,, y_n) + d(x_n, Ty_n) + d(x_n\,, Ty_n)]\ \ (since\ d(x_n\,, Sx_n) \le d(Sx_n\,, Ty_n) + d(x_n\,, Ty_n))$

$=> d(Sx_n\,, Ty_n) \le k\,[\ d(Sx_n\,, Ty_n) + 2\,d(x_n\,, Ty_n)\,]$

$=> (1-k)\,d(Sx_n\,, Ty_n) \le 2k\,d(x_n\,, Ty_n)$

$=> d(Sx_n\,, Ty_n) \le \frac{2k}{1-k}\,d(x_n\,, Ty_n) \to 0\ \ as\ n \to\ \infty$

Therefore $d(Sx_n\,, Ty_n) \to\ 0$

Therefore $Sx_n \to p\ (\,since\ Ty_n\ \to p\,)$

$d(Sx_n\,, Tp) \le k\,[\,d(x_n\,, p) + d(x_n\,, Tp) + d(p, Sx_n)\,]$

On letting $n \to \infty$ we get $Sx_n \to Tp$

Therefore $p = Tp\ \ (\,since\ Sx_n \to p\ and\ \ Sx_n \to Tp\,)$

Therefore $d(Sp, p) = d(Sp, Tp) \le k\,[\,d(p,p) + d(p, Tp) + d(p, Sp)\,]$

$\qquad = k\,[\,d(p,p) + d(p,p) + d(p, Sp)\,]$

$\qquad = k\,[\,d(p, Sp)\,]$

Therefore $(1-k)\,d(p, Sp) \le 0$

Hence $d(p, Sp) = 0$

Therefore p is a common fixed point of S and T.

NOTE:

(i) Theorem 2.11 follows as a corollary to theorem 3.4 since, by theorem 3.3, every cone convex metric space is a cone metric space with convex property.

Thus the result of R. Krishna Kumar et.al.[4] is a corollary of our result.

(ii) Since there are cone metric spaces with convex property which are not cone convex metric spaces (Example 3.2), our result (theorem3.4) is a proper generalization of the result of R. Krishna Kumar et.al. [4].

Open Problem: Obtain sufficient conditions for the convergence of the sequence $\{x_n\}$ of iterates.

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