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GENERALIZATION OF SIERPIŃSKI SPACE

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ABSTRACT

In 1994, F. J. Craveiro de carvalho and D'Azevedo Breda took up the task of generalizing the Sierpiński space and introduced the concept of locally Sierpiński space ([4]). In this paper, we choose a different approach and propose a generalization of Sierpiński space by defining a topology analogous to Sierpiński topology with nested open sets on any arbitrary non-empty set. We then introduce the notion of Special finite generalized Sierpiński space as a special case of generalized Sierpiński space. We investigate some of the properties of the generalized Sierpiński spaces and obtained a formula for the number of finite generalized Sierpiński topologies using Stirling number of the second kind. Finally we show that every special finite generalized Sierpiński space is a D-space.

Keywords: *Sierpiński Space, Generalized Sierpiński Space, compactness, connectedness, separation axioms, D-space*

AMS Subject Classification: *54A10, 54D05, 54D10, 54D30.*

1. INTRODUCTION

Sierpiński spaces play an important role in topological domain theory (see [1], [10] and [11]). Topological domain theory is a generalization of domain theory that includes a wider collection of topological spaces than traditional domain theory. The Sierpiński spaces also find applications in algebraic geometry and have important relations to the theory of computation and semantics.

In 1994, F. J. Craveiro de carvalho and D'Azevedo Breda took up the task of generalizing the Sierpiński space and introduced the concept of locally Sierpiński space ([4]).They call a topological space X locally Sierpiński, if every point $x \in X$ has neighborhood homeomorphic to the Sierpiński space. They have established many interesting properties of locally Sierpiński spaces (see [3], [4], [5] and [6]). Many others such as M. Rostami (see [8] and [9]) extended this work in other directions. However, this generalization of Sierpiński space does not inherit the nested character of open sets of the Sierpiński space; rather it is only the embedding of the idea of Sierpiński space topologically (by defining homeomorphism) at every point of certain topological space. Hence the concept of locally Sierpiński space (as a generalization of Sierpiński space) proposed by them is not global in this sense.

We observe that the inspiration for the generalization of Sierpiński topology comes from the Sierpiński topology itself where the open sets are linearly ordered by set inclusion. Now we propose a generalization of Sierpiński space by defining a topology analogous to Sierpiński topology with nested open sets on any arbitrary non-empty set as follows:

Definition 1.1: Let X be a non-empty set and $\mathfrak{I}$ a collection of some nested subsets of X indexed by a linearly ordered set $(\Lambda, \leq)$ such that every subset of Λ has a maximum and $\mathfrak{I}$ always contains the void set ϕ and the whole set X, i.e.

$$\mathfrak{I} = \{\phi, A_\lambda, X : A_\lambda \subset X, \lambda \in \Lambda\}$$

such that $A_\mu \subset A_\nu$ whenever $\mu \leq \nu$

Then we observe that $\mathfrak{I}$ qualifies as a topology on X. We shall call it the generalized Sierpiński topology and the pair $(X, \mathfrak{I})$ will be called the generalized Sierpiński space.

Definition 1.2: A generalized Sierpiński space $(X, \mathfrak{I}_F)$ is called finite generalized Sierpiński space if the set X is finite. The corresponding topology $\mathfrak{I}_F$ is then called the finite generalized Sierpiński topology.

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If in particular we define a finite generalized Sierpiński topology $\Im_F^{a_1}$ on $X = \{a_1, a_2, \ldots \ldots . a_n\}$ such that

$$\Im_F^{a_1} = \{ \phi, A_{\alpha_k}^{a_1}, X : A_{\alpha_k}^{a_1} \subset X, \alpha_k \in \Lambda \}$$

Where $A_{\alpha_k}^{a_1} = \{a_1, a_2, \ldots \ldots . a_k\}, (1 \leq k \leq n)$ and $A_{\alpha_0}^{a_r} = \phi$, then such a topology will be called a special finite generalized Sierpiński topology. The set $A_{\alpha_k}^{a_1} (= \{a_1\})$ will be called the nucleus of the topology.

2. MAIN RESULTS

The generalized Sierpiński spaces have many interesting properties. We state some of these properties which are easy to observe.

Proposition 2.1: ϕ and X are the only clopen sets of any generalized Sierpiński space.

Proposition 2.2: A generalized Sierpiński topology is compact if and only if it contains a largest proper subset of X .

If the topology does have a largest proper subset of X, then every open cover must contain the whole set X, since the union of all smaller sets does not cover the space. Conversely, if it does not have such a set, then the union of all the proper subsets of X is X itself, and so this will be an open cover with no finite subcover.

All generalized Sierpiński spaces are not compact in general. For example the topology $\Im = \{\phi, (-n, n), \mathbb{R} : (-n, n), n \in \mathbb{N}\}$ is a generalized Sierpiński topology on $\mathbb{R}$ but it is not compact.

Proposition 2.3: Every generalised Sierpiński space is connected.

For a generalised Sierpiński space $(X, \Im)$, there are no disjoint nonempty open sets H and K such that $X = H \cup K$, hence $(X, \Im)$ is a connected space.

Proposition 2.4: Every special finite generalised Sierpiński space is a T_0-space.

Let $X = \{a_1, a_2, \ldots \ldots . a_n\}$ be any finite set and $\Im_\alpha^F$ be a special finite generalised Sierpiński topology defined on X such that

$$\Im_\alpha^F = \{\phi, \{a_i\}, \{a_i, a_{i+1}\}, \ldots \ldots . \{a_i, a_{i+1}, \ldots a_n\}, \{a_i, a_{i+1}, \ldots a_n, a_1\}, \{a_i, a_{i+1}, \ldots a_n, a_1, a_2\} \ldots . \\ \ldots \ldots . . \{a_i, a_{i+1}, \ldots a_n, a_1, a_2 \ldots a_{i-2}\}, X\}$$

Where $a_i \in \{a_1, a_2, \ldots \ldots . a_n\}$

Let $a_s, a_t \in X$ such that $s \neq t$

If $1 \leq s < t < i$, we have $a_s \in \{a_i, a_{i+1}, \ldots a_n, a_1, a_2 \ldots a_s\}$ and $a_t \notin \{a_i, a_{i+1}, \ldots a_n, a_1, a_2 \ldots a_s\}$, i.e. we have an open set containing one point and not the other, i.e. the space $(X, \Im_\alpha^F)$ is a T_0-space.

Using similar arguments the result can be shown to hold for the cases $1 \leq t < s < i$, $i < s < t \leq n$, $i < t < s \leq n$, $i = t < s \leq n$, $i = s < t \leq n$, $s < t = i$ and $t < s = i$ as well.

Note: However this is not true for all finite generalised Sierpiński spaces. For example if $X = \{a_1, a_2, a_3\}$ and $\Im = \{\phi, \{a_1, a_2\}, \{a_1, a_2, a_3\}\}$, then for any two distinct points in X, there is no open set containing one and not the other.

None of the higher separation axioms holds for a generalised Sierpiński space. For example, if we take $X = \{a_1, a_2, a_3\}$ and $\Im = \{\phi, \{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}\}$, we observe that each open set containing a_2 also contains a_1, i.e. there is no open set containing a_2 and not the other point a_1.

Hence the generalised Sierpiński space $(X, \Im)$ does not satisfy the T_1-axiom.

Also since in any generalised Sierpiński space the open sets are nested by the relation of set-inclusion, there are no disjoint open sets, i.e. none of the higher separation axioms (T_2, T_3 or T_4-axiom) is satisfied.

Proposition 2.5: For a given singleton nucleus, a total of $(n-1)!$ special finite generalized Sierpiński topologies can be defined on any n-element finite set and hence the total number of special finite generalized Sierpiński topologies which can be defined on X will be equal to $n!$.

The proposition is easy to be proved.

The problem of enumeration of the topologies on a finite set is studied by Krishnamurthy [7] and others. There is no known simple formula giving $T(n)$; the number of topologies on a finite set with n elements. Now we prove a formula for the number of finite generalized Sierpiński topologies on an n-element set X .

Theorem 2.1: Let $G(n,k)$be the number of finite generalized Sierpiński topologies on an n-element set X having $k(\geq 2)$ non-empty open sets, then

$$G(n,k) = k!\,S(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

Where $S(n,k)$denotes the Stirling numbers of the second kind.

Proof: Let $X = \{a_1, a_2, \dots \dots . a_n\}$ be any finite set and $\mathfrak{I}_\alpha^{a_r}(a_r \in X)$ be a finite generalized Sierpiński topology defined on X such that

$$\mathfrak{I}_\alpha^{a_r} = \{\phi, A_{\alpha_m}^{a_r}, X : A_{\alpha_m}^{a_r} \subset X, \alpha_m \in \Lambda\}$$

Where $a_r \in A_{\alpha_m}^{a_r}$ for $1 \leq m \leq k$ and

$$\phi = A_{\alpha_0}^{a_r} \subset A_{\alpha_1}^{a_r} \subset A_{\alpha_2}^{a_r} \subset \dots \dots .. \subset A_{\alpha_k}^{a_r} = X$$

i.e. $A_{\alpha_1}^{a_r}, A_{\alpha_2}^{a_r}, \dots \dots A_{\alpha_k}^{a_r}$are the k non-empty open sets of $(X, \mathfrak{I}_\alpha^{a_r})$.

There is a bijective correspondence between the k-ordered partitions (partitions having k blocks) of the set $X$and the chains of subsets of X having k non-empty members, i.e. for the chain

$$\phi \neq A_{\alpha_1}^{a_r} \subset A_{\alpha_2}^{a_r} \subset \dots \dots .. \subset A_{\alpha_k}^{a_r} = X$$

the ordered partition $(B_{\alpha_1}^{a_r}, B_{\alpha_2}^{a_r}, \dots \dots . B_{\alpha_k}^{a_r})$ associated with it is given by

$$B_{\alpha_1}^{a_r} = A_{\alpha_1}^{a_r}, B_{\alpha_m}^{a_r} = A_{\alpha_m}^{a_r} - A_{\alpha_{m-1}}^{a_r}, 2 \leq m \leq k$$

On the other hand, if $(B_{\alpha_1}^{a_r}, B_{\alpha_2}^{a_r}, \dots \dots . B_{\alpha_k}^{a_r})$is an ordered partition, then the chain $\phi \neq A_{\alpha_1}^{a_r} \subset A_{\alpha_2}^{a_r} \subset \dots \dots .. \subset A_{\alpha_k}^{a_r} = X$ associated with the partition is given by

$$A_{\alpha_1}^{a_r} = B_{\alpha_1}^{a_r}, A_{\alpha_m}^{a_r} = A_{\alpha_{m-1}}^{a_r} \cup B_{\alpha_m}^{a_r}, 2 \leq m \leq k$$

Now the number of partitions of an n-element set into k blocks is $S(n,k)$. Where $S(n,k)$ is the Stirling number of the second kind given by

$$S(n,k) = S_{n,k} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

Hence the total number of such partitions, i.e. the total number of such chains and hence the total number of finite generalized Sierpiński topologies with $k(\geq 2)$non-empty open sets will be $k!\,S(n,k)$
i.e.

$$G(n,k) = k!\,S(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

Hence the result.

Interestingly some algebra comes into play when we use the following notation for special finite generalized Sierpiński topologies.

$$\mathfrak{I}_f = \mathfrak{I}(t_1, t_2, \dots \dots . t_n) = \{\phi, A_{\alpha_k}^{t_1}, X : A_{\alpha_k}^{t_1} \subset X, \alpha_k \in \Lambda\},$$

Where $f \in S_n$ is a permutation defined on $X = \{a_1, a_2, \dots \dots . a_n\}$ such that $f = \begin{pmatrix} a_1 & a_2 & \dots \dots a_n \\ t_1 & t_2 & \dots \dots . t_n \end{pmatrix}$ and $t_1, t_2, \dots \dots . t_n \in \{a_1, a_2, \dots . a_n\}$.

It is easy to show that the family of all such special finite generalized Sierpiński topologies $\mathcal{T}_X = \{\mathfrak{I}_f : f \in S_n\}$ forms a group under the composition $\star$ given by $\mathfrak{I}_f \star \mathfrak{I}_g = \mathfrak{I}_{fog}$.

Now using the definition of isomorphism of groups we can prove the following proposition-

Proposition 2.6: The group of topologies $(\mathcal{T}_X, \star)$ is isomorphic to the symmetric group S_n

Definition 2.3 ([2]): A subset $D$of a topological space $(X, \mathfrak{I})$ is said to be discrete if every point d in $D$is of the form $\{d\} = G \cap D$ for some open set G.

Definition 2.4 ([12]): A neighborhood assignment on a topological space $(X, \mathfrak{I})$is a function $\varphi: X \longrightarrow \mathfrak{I}$ such that

$$x \in \varphi(x)$$

A topological space $(X, \mathfrak{I})$ is a D-space if for every neighborhood assignment φ on X there is a closed discrete subset $D \subset X$ such that

$$\bigcup_{x \in D} \varphi(x) = X$$

Theorem 2.2 ([2]):

(1) A closed subset of a D-space is a D-space.

(2) If $X = X_1 \cup X_2$ where $X_1$is a closed D-space and X_2 is a D-space, then X is a D-space.

Theorem 2.3: Every special finite generalized Sierpiński space is a D-space.

Proof: Let $X = \{a_1, a_2, \dots \dots . a_n\}$. We shall prove the result by mathematical induction on the number of elements in X denoted by $|X|$.

When $|X| = 1$, i.e. $X = \{a_1\}$, then the special finite generalized Sierpiński topology defined on $X$is given by

$$\mathfrak{I}_F^S = \{\phi, \{a_1\}\}$$

The only neighborhood assignment f on X is the function $f: X \longrightarrow \mathfrak{I}_F^S$ given by

$$f(a_1) = \{a_1\}$$

Now the closed sets of $(X, \mathfrak{I}_F^S)$ are ϕ and $\{a_1\}$. Hence if we take $D = \{a_1\}$ then for the open set $G(= \{a_1\})$, we have

$$G \cap D = \{a_1\} \cap \{a_1\} = \{a_1\}$$

$\Longrightarrow D = \{a_1\}$ is a closed discrete subset of X.

Now

$$X = \{a_1\} = f(a_1)$$

Hence $(X, \mathfrak{I}_F^S)$ is a D-space.

Thus the result is true for $|X| = 1$.

Let us now assume that the result is true for any special finite generalized Sierpiński space $X$with $|X| \leq n - 1$.

Now for $X = \{a_1, a_2, \dots \dots . a_n\}$ the special finite generalized Sierpiński topology $\mathfrak{I}_F^S$ is given by

$$\mathfrak{I}_F^S = \{\phi, \{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\} \dots . \{a_1, a_2, \dots . a_{n-1}\}, X\}$$

Obviously $\{a_n\}$ is a closed subset of X.

Now we can write

$$X = \{a_1, a_2, \dots \dots . a_{n-1}\} \cup \{a_n\}$$

By induction $\{a_1, a_2, \dots \dots .\, a_{n-1}\}$ is a D-space. Hence X is a union of a D-space and a closed D-space. By theorem 2.2 the space $(X, \Im_F^S)$ is a D-space.

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