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STRUCTURES ON Q–FUZZY LEFT H–IDEALS INTERMS OF HEMI RINGS

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ABSTRACT

In this paper we apply the Biswas's idea of Q – fuzzy subgroups to left h-ideals of hemi rings. We introduce the notion of Q – fuzzy subgroups to left h-ideals in hemi rings and investigate some of related properties. Relationship between Q – fuzzy left h – ideals and Q – fuzzy left h – ideals of hemi ring is also given.

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INTRODUCTION

Ideals of hemi rings play a central role in the structure theory and are very useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogues in hemi rings using only ideals. Henriksen defined in [4] a more restricted class of ideals in semi rings, which is called the class of k-ideals, with the property that if the semi ring R is a ring then a complex in R is a k- ideal if and only if it is a ring ideal. Another more restricted, but very important class of ideals, called h- ideals, has been given and investigated by Izuka [5] and La Torre [7] other important results connected with fuzzy ideals in hemi rings were obtained in [6].The concept of Q- fuzzy subgroups can be obtained in [11] [12] [13] [14] . In this paper, we introduced the notion of Q- fuzzy left h-ideals in terms of hemi rings and investigate their properties.

PRELIMINARIES

In this section, we review some elementary aspects that are necessary for this paper.

Definition 2.1: An algebra $(R, +, \cdot)$ is said to be a semi ring if it satisfies the following conditions

$(R, +)$ is a semi group $(R,\cdot)$ is a semi group

$a. (b + c) = a.b + a.c$ and $(b + c).a = b.a + c.a \;\forall\; a, b, c \in R$.

Definition 2.2 : A semi ring $(R, +, \cdot)$ is called a hemiring H_1 : + is commutative and H_2: there exists an element $0\in R$ such that 0 is the identity of $(R,+)$ and the zero element of $(R,\cdot)$ i.e, $0.a = a.0 = 0 \;\forall\; a \in R$. A subset I of a semi ring R is called a left ideal of R if I is closed under addition and $RI \subseteq I$. A left ideal of R is called a left K-ideal of R if $y, z \in I$ and $x\in R$, $x + y = z$ implies $x\in I$.

A left h – ideal of a hemi ring R is defined to be a left ideal A of R such that $(x + a + z = b + z \rightarrow x \in A, \forall(x, z \in R), (\forall a, b \in A))$.

Right h- ideals are defined similarity.

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Definition 2.3: A mapping f: $R_1 \rightarrow R_2$ is said to be hemi ring homomorphism of R_1 is to R_2 if

f(x + y) = f(x) + f(y) and f(xy) = f(x).f(y) for all x, y ∈ R.

Definition 2.4: A mapping μ: X → [0, 1] where X is an arbitrary non – empty set is called a fuzzy set is X. For any fuzzy set μ is X and any X ∈ [0, 1] we defined the set L (μ:α) = {x ∈X / μ(x) ⊆α} which is called lower level cut of μ.

Definition 2.5: Let Q and G be a set and a group respectively. A mapping μ: G × Q → [0, 1] is called a Q – fuzzy set.

Definition 2.6: A fuzzy subset is of a semi ring R is said to be Q - fuzzy left h – ideal of R if
(i) μ(x + y, q) ≥ min{ μ(x, q), μ(y, q)} ∀ x, y ∈ R, q ∈ Q
(ii) μ (xy, q) ≥ μ(y, q) ∀ x, y ∈ R, q ∈ Q

Note that if μ is a Q - fuzzy left h – ideal if a hemi ring R, then μ(0,q) ≥ μ(x, q) ∀ x ∈ R.

Definition 2.7: A fuzzy subset μ of a hemi ring R is said to be an Q - fuzzy left h – ideal of R if
1. μ(x + y ,q) ≥ min{μ(x, q), μ(y, q)} ∀ x, y ∈ R, q ∈ Q
2. μ(xy, q) ≥ μ(y, q) ∀ x, y ∈ R, q ∈ Q
3. x + a + z = b + z → μ(x, q) ≥ min{ μ(a, q), μ(b, q)}

Example: Let R = {0, 1, 2, 3, 4} be a hemi ring with zero multiplication and addition defined by the following table

	0	1	2	3	4
0	0	1	2	3	4
1	1	1	4	4	4
2	2	4	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

We define a fuzzy set μ: R → [0, 1] by letting μ (0) = t_1 and μ (x) = t_2 ∀ x ≠ 0, $t_1 \subset t_2$. By routine computations, we can also easily check that μ is an fuzzy left h –ideal of hemi ring R.

Properties of Q – anti fuzzy left h – ideals

Proposition 3.1 let 'R' be a hemi ring and 'μ' be a Q – fuzzy set in R. Then μ is Q- fuzzy left h – ideal in R if and only if μ^c is a Q-fuzzy left h-ideal in R.

Proof: Let μ be an Q –fuzzy left h – ideal in R. For x, y ∈ R, We have

$$\mu^c(x + y, q) = 1- \mu(x + y, q)$$
$$\leq 1 - \min \{\mu(x, q), \mu(y, q)\}$$
$$= \max \{1 - \mu(x, q), 1- \mu(y, q)\}$$
$$= \max \{\mu^c(x, q), \mu^c(y, q)\}$$

$$\mu^c (xy, q)) = 1- \mu(xy, q)$$
$$\leq 1- \mu(y, q)$$
$$= \mu^c(y, q)$$

Let x, z, a, b ∈R be such that x+ a + z = b + z

$\leq 1 - \min\{\mu(a, q), \mu(b, q)\}$

$= \max\{1 - \mu(a, q), 1- \mu(b, q)\}$

$= \max\{\mu^c(a ,q), \mu^c(b, q)\}$ Hence μ^c is a Q- fuzzy left h – ideal of R .

Conversely, μ^c is a Q - fuzzy left h – ideal of R. For x, y ∈R, q∈ Q, we have

$$\begin{aligned}
\mu(x + y, q) &= 1- \mu^c(x + y, q) \\
&\geq 1 - \max\{\mu^c(x ,q), \mu^c(y, q)\} \\
&= \min\{\mu(x ,q), \mu(y, q)\} \\
\mu(xy, q) &= 1- \mu^c(xy, q) \\
&\geq 1- \mu^c(y, q) \\
&= \mu(y, q)
\end{aligned}$$

Let x, z, a, b ∈R be such that x+ a + z = b + z,

Then $\mu(x ,q) = 1- \mu^c(x ,q)$

$\leq 1 - \max\{\mu^c(a ,q), \mu^c(b, q)\}$

$= \min\{\mu(a ,q), \mu(b, q)\}$

$= \min\{\mu(a, q), \mu(b, q)\}$

Hence μ is a Q- fuzzy left h – ideal of R.

Proposition 3.2 let 'μ' be Q – fuzzy left h – ideal in a hemi ring R such that $L(\mu:\alpha)$ is a left h – ideal of R for each $\alpha\in I_m(\mu)$, $\alpha\in[0,1]$.Then μ is Q- fuzzy left h-ideal in R.

Proof: Let x, y ∈ R, q ∈ Q be such that $\mu(x, q) = \alpha_1,$ $\mu(y, q) = \alpha_2$ then x + y ∈ L (μ:α) without loss of generality, we may assume that $\alpha_1 > \alpha_2.$ It follows that $L(\mu;\alpha_2)\subseteq L(\mu;\alpha_1)$ so that $x \in L(\mu;\alpha_1)$ and $y \in L(\mu;\alpha_2)$.Since $L(\mu;\alpha_1)$ is a left is a left h – ideal of R. We have $x + y \in L(\mu; \alpha_1)$. Thus

$\mu(x + y, q) \geq \alpha_1 = \min\{\mu(x ,q), \mu(y, q)\}$

$\mu(xy, q) \geq \alpha_1 = \mu(y, q)$

Let x, z, a, b ∈R be such that x+ a + z = b + z, then $\mu(x, q) \geq \alpha_1 = \max\{\mu(a, q), \mu(b, q)\}$.

This shows that μ is an Q- fuzzy left h-ideal in R.

Corollary 3.3: Let μ be Q-fuzzy left h-ideal in R then μ is an Q- fuzzy left h-ideal in R if and only if L(μ; α) is a left h-ideal in R for every $\alpha\in[\mu(0,q),1]$ with $\alpha\in[0,1]$.

Proposition 3.4 : let 'μ' be Q – fuzzy set in a hemi ring R then two lower level subsets $L(\mu;t_1)$ and $L(\mu;t_2)$, $(t_1 < t_2)$ are equal iff there is no x ∈ R such that $t_1 \leq \mu(x, q) \leq t_2.$

Proof: From definition of L (μ; α) it follows that $L(\mu; t) = \mu^{-1}([\mu(0, q); t])$ for t∈ [0, 1]. Let $t_1, t_2\in [0, 1]$ be such that $t_1 < t_2$ then $L(\mu ; t_1) = L(\mu ; t_2)$

$\Leftrightarrow \mu^{-1}([\mu(0,q);t_1]) = \mu^{-1}([\mu(0,q);t_2])$

$\Leftrightarrow \mu^{-1}(t_1, t_2) = \varphi$

⇔ There is no x ∈R such that $t_1 \leq \mu(x, q) \leq t_2.$ This completes the proof.

Definition 3.5: A left h – ideal A of hemi ring R is said to be characteristic iff f (A) = A, for all f ∈ Aut (R), Where Aut (R) is the set of all automorphisms of R. Q – fuzzy left h – ideal μ of hemi ring R is said to be Q – fuzzy characteristic if μ^f (x, q) = μ(x, q) for all and f ∈ Aut (R).

Lemma3.6: Let μ be an Q - fuzzy left h – ideal of a hemi ring R and let x ∈ R then μ(x ,q) = s iff x ∈ L(μ ; s) and x ∉ L(μ ; t) for all s > t.

Proof: Straight forward.

Proposition 3.7: let 'μ' be an Q – fuzzy left h – ideal of a hemi ring R then each

level left h – ideal of μ is characteristic iff μ is an Q- fuzzy Characteristic.

Proof: Suppose that μ is an Q- fuzzy Characteristic and let S ∈ I_m(μ), f ∈ Aut(R) and x ∈ L(μ ; s) then
μ^f (x ,q) = μ(x ,q) ≥ s ⇒ μ(f(x ,q)) ≥ s ⇒ f(x ,q) ∈ L(μ ; s)

Thus

f(L(μ ; s)) and y ∈R such that f(y ,q) = (x ,q) then μ(y ,q) = μ^f (y ,q) = μ(f(y ,q)) = μ(x ,q) ≥ s
⇒ y ∈ L(μ : s) so that (x ,q) = f(y ,q) ∈ L(μ ; s). Consequently, L (μ; s) ⊆ f (L (μ; s)).

Hence f (L (μ; s)) = L (μ; s)

i.e. L(μ ; s) is characteristic. Conversely, suppose that each level h – ideal of μ is characteristic and let x ∈ R, f ∈ Aut (R) and μ(x ,q) = s . then by virtue Lemma(3.6) x ∈ L(μ ; s) and x ∉ L(μ ; t) for all s > t. It follows from the assumption that

f(x ,q) ∈ f (L(μ ; s))= L(μ) so that μ^f (x ,q) = μ(f(x ,q)) ≥ s .

Let t = μ^f (x ,q) and assume that s > t. then f(x, q) ∈ L(μ ; t)= f(L(μ ; t)) which implies from the injectivity of f that x ∈ L(μ : t), a contradiction.

Hence μ^f (x, q) = μ (f(x, q)) = s = μ(x, q) showing that μ is an Q – anti fuzzy characteristic.

Proposition 3.8: Let f: $R_1 \to R_2$ be an epimorphism of hemi rings. If V is an Q – fuzzy left - ideal of R_2 and μ is the pre – image of V under f then μ is an Q- fuzzy left h-ideal of R_1.

Proof: For any x, y ∈ R_1 and q ∈Q, we have

$$\begin{aligned} \mu(x + y, q) &= V(f(x + y, q)) \\ &= V(f(x ,q) + f(y, q)) \\ &\geq \min \{V(f(x ,q), V(f(y, q))\} \\ &= \min \{ \mu(x ,q), \mu(y, q)\} \\ \mu(xy, q) &= V(f(xy, q)) \\ &= V(f(x ,q). f(y, q)) \\ &\geq V(f(x ,q)) \\ &= \mu(y, q) \end{aligned}$$

Let x, z, a, b ∈R be such that x+ a + z = b + z, then we have

$$\begin{aligned} \mu(x, q) &= V(f(x, q)) \\ &\geq \min \{V(f(a ,q)), V(f(b, q))\} \end{aligned}$$

$= \min\{\mu(a, q), \mu(b, q)\}$

Hence μ is an Q – fuzzy left h- ideal of R_1.

Definition3.9: Let R_1 and R_2 be two hemi rings and f be a function of R_1 into R_2. If μ is a Q – fuzzy in R_2 then the Pre–image of μ under f then μ is the Q – fuzzy in R_1 defined by

$f^{-1}(\mu)(x, q) = \mu(f(x, q))\ \forall\ x \in R_1,\ q \in Q$.

Proposition 3.10: Let f: $R_1 \to R_2$ be an onto homomorphism of hemi rings. If μ is an Q –fuzzy left h- ideal of R_2 then $f^{-1}(\mu)$ is an Q –anti fuzzy left h- ideal of R_1

Proof: Let $x_1, x_2 \in R_1$, then we have

$$\begin{aligned} f^{-1}(\mu)(x_1+x_2,q) &= \mu(f(x_1,q) + f(x_2,q)) \\ &\geq \min\{\mu(f(x_1,q)) + \mu(f(x_2,q))\} \\ &= \min\{f^{-1}(\mu)(x_1,q), f^{-1}(\mu)(x_2,q)\} \end{aligned}$$

$$\begin{aligned} f^{-1}(\mu)(x_1x_2,q) &= \mu(f(x_1, q) f(x_2, q)) \\ &\geq \mu(f(x_2, q)) \\ &= f^{-1}(\mu)(x_2, q) \end{aligned}$$

Let x, z, a, b $\in R_1$ be such that x+ a + z = b + z, then we have

$$\begin{aligned} f^{-1}(\mu)(x, q) &= \mu(f(x, q)) \\ &\geq \min\{\mu(f(a,q)), \mu(f(b,q))\} \\ &= \min\{f^{-1}(\mu)(a, q), f^{-1}(\mu)(b, q)\} \end{aligned}$$

Hence $f^{-1}(\mu)$ is an Q – fuzzy left h- ideal of R_1

Definition3.11: Let R_1 and R_2 be any sets and let f: $R_1 \to R_2$ be any function. A be Q – fuzzy subset μ of R_1 is called f- invariant if f(x) = f(y) implies $\mu(x, q) = \mu(y, q))\ \forall\ x, y \in R,\ q \in Q$.

Proposition 3.12: Let f: $R_1 \to R_2$ be an epimorphism of hemi rings. Let μ be an f – invariant Q – fuzzy left h- ideal of R_1, then f (μ) is an Q – fuzzy left h- ideal of R_2.

Proof: Let x, y$\in R_2$ then there exists a, b $\in R_1$, such that f(a) = x and f(b) = y then x + y = f (a + b) and xy = f(ab).

Since μ is invariant,

$$\begin{aligned} f(\mu)(x + y, q) &= \mu(x + y, q) \\ &\geq \min\{\mu(a, q)), \mu(b, q))\} \\ &= \min\{f(\mu)(x, q), f(\mu)(y, q)\} \end{aligned}$$

$$\begin{aligned} f(\mu)(xy, q) &= \mu(ab, q)) \\ &\geq \mu(b, q)) \\ &\geq f(\mu)(y, q) \end{aligned}$$

Let x, z, a, b $\in R_2$ be such that x+ a + z = b + z, then there exist $\overline{x}, \overline{z}, \overline{a}, \overline{b}$ such that $f(\overline{x}) = x$, $f(\overline{y}) = y$, $f(\overline{a}) = a$, $f(\overline{b}) = b$, since μ is f- invariant,

$f(\mu)(x, q) = \mu(x, q)$

$\geq \min \{\mu\ (a\ ,q)), \mu\ (b\ ,q)\}$

$= \min \{f(\mu)(a\ ,q), f(\mu)(b, q)\}$

Hence f (μ) is an Q – fuzzy left h- ideal of R_2.

Definition 3.13: A Q- fuzzy left h-ideal μ of a hemi ring R is said to be normal if there exist x ∈R such that μ(x, q) = 1. Note that if μ is a normal Q – anti fuzzy left h – ideal of R_1 then μ (0, q) = 1 and hence μ is normal if and only if μ (0, q) = 1.

Proposition 3.14: Let μ be an Q – fuzzy left h – ideal of a hemi ring R. Let μ^+ be a Q – fuzzy set in R defined by $\mu^+(x, q) = \mu(x, q) = 1 - \mu(0,q)$ for all x ∈R then μ^+ is a normal Q – fuzzy left h – ideal of R which contains μ .

Proof: For any x, y ∈ R, we have

$\mu^+\ (x, q) = \mu(0,q) + 1- \mu(0,q) = 1$

And $\mu^+ (x+y, q) = \mu (x+y, q) + 1- \mu(0,q)$

$\geq \min \{(\mu\ (x ,q), \mu\ (y, q)) + 1- \mu(0,q) \}$

$= \min \{(\mu (x ,q) + 1- \mu(0,q)), (\mu\ (y ,q) + 1- \mu(0,q))\}$

$= \min \{\mu^+ (x, q), \mu^+ (y, q)\}$

$\mu^+ (xy, q) = \mu (xy, q) + 1- \mu(0,q)$

$\geq \mu\ (y, q) + 1- \mu(0,q)$

$= \mu^+\ (y, q)$

This Shows that μ^+ is an q – fuzzy left h – ideal of R. Let a, b, x, z ∈ R be such that x+ a + z = b + z, then

$\mu^+ (x, q) = \mu\ (x ,q) + 1- \mu(0,q)$

$\geq \min \{ (\mu\ (a ,q)), \mu\ (b, q)) + 1- \mu(0,q)\}$

$= \min \{(\mu (a, q) + 1- \mu (0, q)), (\mu (b, q) + 1- \mu (0, q))\}$

$= \min \{\mu^+ (a, q), \mu^+ (b, q)\}$

Hence μ^+ is a normal Q – anti fuzzy left h – ideal of hemi ring of R. Clearly $\mu \leq \mu^+$.

Definition 3.15: Let N(R) denote the set of all normal Q –fuzzy left h – ideals of R. Note that N(R) is a Poset under the set inclusion. A Q – fuzzy set μ in a hemi ring R is called a Maximal Q –fuzzy left h – ideal of R if it is non – constant and μ^+ is a maximal element of (N(R), ⊆).

Proposition 3.16: Let μ ∈N(R) be non – constant such that it is a maximal element of (N(R),⊆) then it takes only two values {0,1}.

Proof: Since μ is normal, μ(0,q) = 1. We claim that μ(x, q) = 0. If not, then there exists x_0 ∈R such that $0 < \mu(x_0,q) < 1$. Define on R a Q – fuzzy set V by putting $V(x,q) = ½ \{ \mu (x ,q) + \mu(x_0,q)\}$ for each x ∈R then clearly V is well defined and for all x, y ∈R. we have

$V(x+y, q) = ½ \mu (x+y, q) + ½ \mu(x_0, q)$

$\geq ½ \{\min\{\mu\ (x ,q), \mu\ (y ,q)\}+ \mu(x_0, q)\}$

$= \min \{V(x, q), V(y, q)\}$

$V(xy, q) = ½ μ (xy ,q) + ½ μ(x_0,q)$

$\geq ½ (μ (y ,q)+ μ(x_0,q))$

$= V(y, q)$

Thus V is an Q –fuzzy left h- ideal of R.

Let a, b, x, z ∈R be such that x+ a + z = b + z, then

$V(x, q) = ½ μ (x, q) + ½ μ(x_0, q)$

$\geq ½ \{min \{μ (a, q), μ (b, q)\} + μ(x_0, q)\}$

$= min \{½(μ (a, q) + μ(x_0, q), ½(μ (b ,q) + μ(x_0,q)\}$

$= min \{V (a, q), V (b, q)\}$

Hence V is an Q-anti fuzzy left h – ideal of R. By theorem 3.16, V^+ is a maximal Q- anti fuzzy left h –ideal of R. Note that

$V^+ (x, q) = V (x , q) + 1- V(0,q)$

$= ½(μ (x, q) + μ(x_0, q) + 1- ½(1+ μ(x_0, q))$

$= ½(μ(x_0, q) +1)$

$= V(x_0, q)$

and $V^+ (x_0,q) \geq 1 = V^+ (0,q)$

Hence V^+ is a non – constant and μ is not a maximal non - constant and μ is not a maximal element of N(R). This is a contradiction.

CONCLUSION

Y. B. Jun [6] introduced the concept on fuzzy h- ideals in hemi rings and [14] investigated the idea of anti fuzzy left h-ideals in hemi rings. In this paper, we established some structure properties of Q- fuzzy left h- ideals in a hemi rings. One can obtain similar results, by using the intuitionist anti fuzzy ideals in a hemi rings.

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