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COFINITELY WEAK RAD-SUPPLEMENTED MODULES

S. K. Choubey*, B. M. Pandeya, A. J. Gupta† and H. Ranjan‡**

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi-221005, India

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ABSTRACT

Let R be a ring and M be an R-module. M is called cofinitely weak Rad-supplemented module if every cofinite submodule of M has a weak Rad-supplement in M. If every cofinite submodule of M has ample weak Rad-supplements in M, then M is called amply cofinitely weak Rad-supplemented module. In this paper we study some properties of such type of modules.

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Key words: *Cofinitely supplemented module, Cofinitely Rad-supplemented module, cofinitely weak Rad-supplemented module, amply cofinitely weak Rad-supplemented module*

1. INTRODUCTION

Throughout this paper all rings are associative rings with identity and all modules are unital left R-modules. Let R be a ring and M be an R-module. The notation $N \subseteq M$ means that N is a submodule of M. A submodule K of an R-module M is called small in M (denoted by $K \ll M$) if $K + L = M$ for any submodule L of M implies $L = M$, see [1]. $RadM$ indicates the Jacobson radical of M. A module M is called semi-hollow if every finitely generated proper submodule is small in M, or $RadM = M$, see [2]. Let M be an R-module and A and B be any submodules of M. B is called a supplement of A in M if B is minimal with respect to $M = A + B$. B is a supplement of A in M iff $M = A + B$ and $A \cap B \ll B$, (see [2] 20.1). M is called supplemented if every submodule of M has a supplement in M. Artinian and semisimple modules are supplemented modules. For $A \subseteq M$, a submodule B of M is called a weak supplement of A in M if $A + B = M$ and $A \cap B \ll M$ (see [12], 1.3). An R-module M is called weakly supplemented if every submodule of M has a weak supplement in M. Clearly supplemented modules are weakly supplemented. Artinian, semisimple and hollow modules are weakly supplemented modules.

A submodule K of a module M is said to be cofinite if the factor module M/K is finitely generated. If every cofinite submodule of M has a supplement in M then M is called a cofinitely supplemented module, see [3]. An R-module M is called a cofinitely weak supplemented module (or briefly a cws-module) if every cofinite submodule has a weak supplement. Clearly cofinitely supplemented module and weakly supplemented module are cofinitely weak supplemented and a finitely generated module is weakly supplemented iff it is a cws-module. Cofinitely weak supplemented module need not be cofinitely supplemented. For example, consider the ring $R = Z_{p,q} = \{a/b : a,b \in Z, b \neq 0, (p,b) = 1, (q,b) = 1\}$, the left module ${}_R R$ is cofinitely weak supplemented but is not cofinitely supplemented; see [4, remark 3.3].

Let M be an R-module and let U be a submodule of M. A submodule V of M is called a Rad-supplemented of U in M (according to [5], generalized supplement) if $U + V = M$ and $U \cap V \subseteq RadV$. An R-module M is called Rad-supplemented (according to [5], generalized supplemented or a GS-module) if every submodule of M has a Rad-supplement in M. A submodule V of M is called a weak Rad-supplement of U in M if $U + V = M$ and $U \cap V \subseteq RadM$. An R-module M is called weakly Rad-supplemented (according to [5], generalized weakly supplemented or a WGS-module) if every submodule of M has a weak Rad-supplement in M. The Z-module Q is Rad-supplemented as well as weak Rad-supplemented modules but the Z-module Q is not supplemented. Let M be an R-module. If every cofinite submodule of M has a Rad-supplement in M then M is called a cofinitely Rad-supplemented module.

***Corresponding author:** S. K. Choubey*, *Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi-221005, India*

Let M be an R-module and $N \in \sigma[M]$, subcategory of left R-modules subgenerated by M. A projective module P in $\sigma[M]$ together with a small epimorphism $\pi : P \to N$ is called a projective cover of N in $\sigma[M]$. A module N in $\sigma[M]$ is called semiperfect in $\sigma[M]$ if every factor module of N has a projective cover in $\sigma[M]$. A projective module in $\sigma[M]$ is semiperfect in $\sigma[M]$ if and only if it is (amply) supplemented (see [1], 42.3).

2. COFINITELY WEAK RAD-SUPPLEMENTED MODULE

In this section, we discuss the concept of cofinitely weak Rad-supplemented modules and give some properties of such type of modules.

Definition 2.1. Let M be an R-module. M is called a cofinitely weak Rad-supplemented module if every cofinite submodule of M has a weak Rad-supplement in M.

Lemma 2.2. Let M be an R-module and V be a weak Rad-supplement of U in M. Then $(V + L)/L$ is a weak Rad-supplement of U/L in M/L for every submodule L of U.

Proof: See [6, Lemma II. 1]

Theorem 2.3. Let M be an R-module and N be a nonzero semi-hollow submodule of M. Then M is cofinitely weak Rad-supplemented iff M/N is cofinitely weak Rad-supplemented.

Proof: Let M be a cofinitely weak Rad-supplemented module. Let U be a submodule of M and N is a nonzero semi-hollow submodule of M. Consider U/N is a cofinite submodule of M/N, then U is cofinite. Since M is cofinitely weak Rad-supplemented module, then there is a submodule V of M such that $U + V = M$ with $U \cap V \subseteq RadM$. By lemma 2.2, $(V + N)/N$ is a weak Rad-supplement of U/N in M/N. Hence M/N is cofinitely weak Rad-supplemented.

Conversely, Let U be a cofinite submodule of M. Then $(U + N)/N$ is a cofinite submodule of M/N. Since M/N is cofinitely weak Rad-supplemented, $(U + N)/N$ has a weak Rad-supplement in M/N. Suppose V/N is weak Rad-supplement of $(U + N)/N$ in M/N. Then $V/N + (U + N)/N = M/N \Rightarrow (U + V)/N = M/N \Rightarrow U + V = M$ and $V/N \cap (U + V)/N \subseteq Rad(M/N) \Rightarrow (U \cap V)/N \subseteq RadM/RadN \Rightarrow (U \cap V)/N \subseteq RadM/N$ (since N is semi-hollow module, so $RadN = N$) $\Rightarrow U \cap V \subseteq RadM$. Hence M is a cofinitely weak Rad-supplemented.

Proposition 2.4. Suppose that M be a cofinitely weak Rad-supplemented module and N be a submodule with $RadM \subseteq N$. If $Rad(M/N) = \{N\}$, then every cofinite submodule of M/N is a direct summand of M/N.

Proof: Let M be a cofinitely weak Rad-supplemented module and M/N be any factor module of M. For $N \subseteq K$, let K/N be a cofinite submodule of M/N, then $\frac{M/N}{K/N}$ is finitely generated. Now $M/K \cong \frac{M/N}{K/N}$. therefore, M/K is finitely generated. Hence K is a cofinite submodule of M. Since M is a cofinitely weak Rad-supplemented module, therefore, there is a submodule V of M such that $K + V = M$ and $K \cap V \subseteq RadM$. According to the lemma 2.2, $(V + N)/N$ is a weak Rad-supplement of K/N in M/N. Hence $K/N + (V + N)/N = M/N \Rightarrow (K + V)/N = M/N \Rightarrow K + V = M$ and $(V + N)/N \cap K/N \subseteq Rad(M/N) = \{N\}$. Since $RadM \subseteq N$, we know $\{N\} \subseteq (V + N)/N \cap K/N$, therefore we have $(V + N)/N \cap K/N = \{N\}$. Hence K/N is a direct summand of M/N.

Corollary 2.5. Let M be a cofinitely weak Rad-supplemented module. Then every cofinite submodule of $M/RadM$ is a direct summand of $M/RadM$.

Lemma 2.6. If $f : M \to N$ is a homomorphism and a submodule L of M containing $\ker f$ is a weak Rad-supplement in M, then $f(L)$ is a weak Rad-supplement in $f(M)$.

Proof: Let M, N be R-modules and $f: M \to N$ be a homomorphism. If L is a weak Rad-supplement of K in M, then we have $M = L + K \Rightarrow f(M) = f(L) + f(K)$ and since $L \cap K \subseteq RadM$ we have $f(L \cap K) \subseteq f(RadM)$ $\subseteq Rad(f(M))$. As $\ker f \subseteq L$, $f(L) \cap f(K) = f(L \cap K)$ i.e. $f(L) \cap f(K) \subseteq Rad(f(M))$. So $f(L)$ is a weak Rad-supplement of $f(K)$ in $f(M)$.

Proposition 2.7. Every homomorphic image of cofinitely weak Rad-supplemented module is a cofinitely weak Rad-supplemented module.

Proof: Suppose that $f: M \to N$ be a homomorphism and M be a cofinitely weak Rad-supplemented module. Let K is a cofinite submodule of $f(M)$, then $M / f^{-1}(K) \cong (M / \ker f)/(f^{-1}(K))/\ker f \cong f(M)/K$. Therefore, $M / f^{-1}(K)$ is finitely generated. Since M is a cofinitely weak Rad-supplemented module, $f^{-1}(K)$ is a weak Rad-supplemented module in M and according to the lemma 2.6, $K = f(f^{-1}(K))$ is a weak Rad-supplement in $f(M)$.

Corollary 2.8. Any factor module of a cofinitely weak Rad-supplemented module is a cofinitely weak Rad-supplemented module.

3. AMPLY COFINITELY WEAK RAD-SUPPLEMENTED MODULES

In this section, we show the concept of amply cofinitely weak Rad-supplemented modules and give some properties of such type of modules.

Definition 3.1. Let M be an R-module. If every cofinite submodule of M has ample weak Rad-supplements in M then M is called amply cofinitely weak Rad-supplemented module.

Proposition 3.2. Every factor module of an amply cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

Proof: Let M be an amply cofinitely weak Rad-supplemented module. For $A \subseteq X \subseteq M$, let M / A be any factor module of M and X/A be a cofinite submodule of M/A, then $\frac{M/A}{X/A}$ is finitely generated. Now $M/X \cong \frac{M/A}{X/A}$, M/X is also finitely generated. Hence X is a cofinite submodule of M. Suppose $X/A + Y/A = M/A$ for some submodule Y/A of M/A, then $X + Y = M$. Since X is cofinite and M is amply cofinitely weak Rad-supplemented, there is a submodule B of Y such that B is a weak Rad-supplement of X in M. By lemma 2.2, $(B+A)/A$ is a weak Rad-supplement of X/A in M/A. Clearly $(B+A)/A \subseteq Y/A$. Hence M/A is amply cofinitely weak Rad-supplemented.

Corollary 3.3. Every homomorphic image of an amply cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

Proof: Let M be an amply cofinitely weak Rad-supplemented module. Since every homomorphic image of M is isomorphic to a factor module of M, then by proposition 3.2, every homomorphic image of M is amply cofinitely weak Rad-supplemented.

The R-module M is called π-projective, if for every submodules U and V with $M = U + V$ there exists a homomorphism $f: M \to M$ such that $\mathrm{Im} f \subseteq U$ and $\mathrm{Im}(1-f) \subseteq V$ [see, 2].

Proposition 3.4. Let M be a cofinitely weak Rad-supplemented and π-projective module. Then M is amply cofinitely weak Rad-supplemented.

Proof: Let A be a cofinite submodule of M and $A + B = M$ for any submodule B of M. Since M is cofinitely weakly Rad-supplemented and A is a cofinite submodule of M, there exists a weak Rad-supplement T of A in M. Since M is π-projective, there exists an homomorphism $f: M \to M$ such that $\mathrm{Im} f \subseteq B$ and $\mathrm{Im}(1-f) \subseteq A$. Then we can show $f(A) \subseteq A$ and $(1-f)(B) \subseteq B$. In this case

$$M = f(M) + (1-f)(M) = A + f(A+T) = A + f(A) + f(T) = A + f(T).$$

Let $a \in A + f(T)$. Then there exists $t \in T$ with $a = f(t)$. This case $t - a = t - f(t) = (1-f)(t) \in A$ and then $t \in A$. Hence $t \in A \cap T$ and $A \cap f(T) \subseteq f(A \cap T)$. By the Hypothesis, since $A \cap T \subseteq RadM$ so $f(A \cap T) \subseteq f\left(RadM\right)$. $A \cap f(T) \subseteq f(A \cap T) \subseteq f(RadM) \subseteq Rad(f(M)) \subseteq RadM$. Hence $f(T)$ is a weak Rad-supplement of A in M. Since $f(T) \subseteq B$, A has ample weak Rad-supplements in M. Thus M is amply cofinitely weak Rad-supplemented.

Corollary 3.5. Every projective and cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

Proof: We can show that every projective module is π-projective module. Now by the proposition 3.4, every projective and cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

Lemma 3.6. Let M be an R-module with small radical and $A \subseteq M$. If A has a weak Rad-supplement that is a supplement in M, then A has a supplement in M.

Proof: Let B be a weak Rad-supplement of A in M, then $A \cap B \subseteq RadM \ll M$ and so $A \cap B \ll M$. Since B is a supplement in M, $A \cap B \ll B$. Hence B is a supplement of A in M.

Theorem 3.7. Let M be an R-module with small radical. If M is amply cofinitely weak Rad-supplemented such that weak Rad-supplements are supplements in M, then M is amply cofinitely supplemented.

Proof: For proof see lemma 3.6.

Corollary 3.8 Let R be any ring. If the R-module R is weak Rad-supplemented such that weak Rad-supplements are supplements in R, then R is semiperfect.

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