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INVERSE EQUITABLE DOMINATION IN GRAPHS

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ABSTRACT

Let $G=(V,E)$ be a graph with no isolated vertex. A classical observation in domination theory is that if D is a minimum dominating set of G, then $V-D$ is also a dominating set of G. A set D' is an inverse dominating set of G if D' is a dominating set of G and $D' \subseteq V-D$ for some minimum dominating set D of G. The inverse domination number of G is the minimum cardinality among all inverse dominating sets of G. In this paper, we introduce the the equitable inverse domination in a graph and begin an investigation of this concept, some of the properties and interesting results of this new parameter are obtained.

***Keywords**: Dominating set, Equitable dominating set, Inverse equitable dominating set, Inverse equitable domination number.*

***Mathematics Subject Classification:** 05C69.*

1. INTRODUCTION

All graphs in this paper will be finite and undirected, without loops and multiply edges. As usual $p=|V|$ and $q=|E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively. A set D of vertices in a graph G is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. If G is connected graph, then a vertex cut of G is a subset R of V with the property that the subgraph of G induced by $V-R$ is disconnected. If G is not a complete Graph, then the vertex connectivity number $k(G)$ is the minimum cardinality of a vertex cut. If G is complete graph K_p it is known that $k(G)=p-1$.

For terminology and notations not specifically defined here we refer reader to [3]. For more details about domination number and its related parameters, we refer to [4] and [5].

A dominating set S of G is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected the minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$.

A dominating set S of G is called non-split dominating set if the induced subgraph $\langle V-S \rangle$ is connected the minimum cardinality of a non-split dominating set of G is called the non split domination number of G and is denoted by $\gamma_{ns}(G)$.

A dominating set S of G is called total dominating set if the induced subgraph $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$ [4].

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A subset D of $V(G)$ is called an equitable dominating set of a graph G if for every $v \in (V-D)$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called an equitable domination number of G. D is minimal if for any vertex $u \in D$, $D - \{u\}$ is not an equitable dominating set of G. If a vertex $u \in V$ be such that $|d(u) - d(v)| \geq 2$ for all $v \in N(u)$ then u is in equitable dominating set. Such vertices are called equitable isolated vertices. The equitable neighbourhood of u denoted by $N_e(u)$ is defined as $N_e(u) = \{v \in N(u), |deg(u) - deg(v)| \leq 1\}$. The cardinality of $N_e(u)$ is denoted by $deg_e(u)$. The maximum and minimum equitable degree of a vertex in G are denoted respectively by $\Delta_e(G)$ and $\delta_e(G)$. That is $\Delta_e(G) = max_{u \in V(G)} |N_e(u)|$, $\delta_e(G) = min_{u \in V(G)} |N_e(u)|$.

A subset S of V is called an equitable independent set, if for any $u \in S$, $v \notin N_e(u)$, for all $v \in S - \{u\}$.

Let $G = (V, E)$ be a graph with no isolated vertex. A classical observation in domination theory is that if D is a minimum dominating set of G, then $V - D$ is also a dominating set of G. A set D' is an inverse dominating set of G if D' is a dominating set of G and $D' \subseteq V - D$ for some minimum dominating set D of G. The inverse domination number of G is the minimum cardinality among all inverse dominating sets of G. In this paper, we introduce the the equitable inverse domination in a graph and begin an investigation of this concept, some of the properties and interesting results of this new parameter are obtained.

2. INVERSE EQUITABLE DOMINATION NUMBER

Definition. Let $G = (V, E)$ be a graph with no equitable isolated vertices. If D is a minimum equitable dominating set of G, then $V - D$ is also a equitable dominating set of G. A set D' is an inverse equitable dominating set of G if D' is an equitable dominating set of G and $D' \subseteq V - D$ for some minimum equitable dominating set D of G. The inverse equitable domination number of G is the minimum cardinality among all inverse equitable dominating sets of G.

Definition. Let $G = (V, E)$ be a graph with no equitable isolated vertices. If D is minimum dominating set and D' is inverse equitable dominating set with respect to D. Then D' is called minimal inverse equitable dominating set if no proper subset of is an equitable dominating set of G.

Definition. An inverse equitable dominating set D' is called connected inverse equitable dominating set of $G = (V, E)$ if $\langle V - D' \rangle$ is connected.

Example 2.1 *Let G be the graph in the Figure 1, $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. There is only two minimum dominating set v_4, v_5, v_2, v_7 and obviously the minimum inverse dominating set corresponding to v_4, v_5 is v_2, v_7 and visa versa. Therefore $\gamma(G) = 2$ and $\gamma^{-1}(G) = 2$.*

There is only one minimum equitable dominating set v_2, v_7 and there are two corresponding minimum inverse equitable dominating set $\{v_1, v_3, v_4, v_6, v_1\}$ and $\{v_1, v_3, v_5, v_6, v_8\}$. Thus $\gamma_e(G) = 2$ and $\gamma_e^{-1}(G) = 5$.

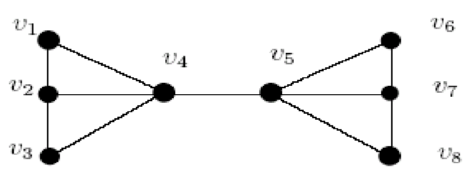

Figure 1: G

Note that every graph without equitable isolated vertices contains an inverse equitable dominating set, since if D is any minimal dominating set then $V - D$ is also equitable dominating set. So here by graph we mean graph without any isolated vertices.

First we obtain the exact value of inverse equitable domination of some standard graphs. Theproof of the following propositions is straightforward.

Proposition 2.2. *For any cycle* C_p *with* p *vertices,* $\gamma_e^{-1}(C_p) = \lceil \frac{p}{3} \rceil$

Proposition 2.3. *For any path* P_p *with* p *vertices,*

$$\gamma_e^{-1}(P_p) = \begin{cases} \lceil \frac{p}{3} \rceil + 1, & p \equiv (0 mod\ 3); \\ \lceil \frac{p}{3} \rceil, & otherwise. \end{cases}$$

Proposition 2.4. *For any complete graph* G *,* $\gamma_e^{-1}(K_p) = 1$.

Proposition 2.5. *For any complete bipartite graph* $K_{m,n}$ *without any equitable isolated vertex vertices,*

$$\gamma_e^{-1}(K_{m,n}) = 2.$$

Proposition 2.6. *For any graph* G *without any equitable isolated vertices,*

$$\gamma_e(G) \le \gamma_e^{-1}(G).$$

Proposition 2.7. *For any graph* G *without any equitable isolated vertices,*

$$\gamma_e(G) + \gamma_e^{-1}(G) \le p.$$

Obviously any minimum equitable dominating set is also minimal but the converse is not true see the following example.

Example. Let G be a graph as in Figure 2. There are two minimum equitable dominating sets $\{v_2, v_7\}$ and $\{v_4, v_5\}$, and there are two minimum inverse equitable dominating sets $\{v_2, v_7\}$ and $\{v_4, v_5\}$.
obviously $\{v_1, v_3, v_6, v_8\}$ is minimal inverse equitable dominating but not minimum inverse equitable dominating set.

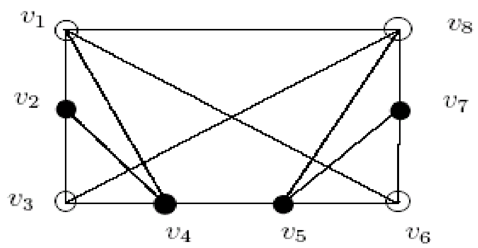

Figure 2: Minimal inverse equitable dominating set

In the following theorem we get the sufficient and necessary condition for the inverse equitable dominating set to be minimal.

Theorem 2.8. *An inverse equitable dominating set* D' *of a graph* G *is minimal if and only if for every vertex* $u \in D'$ *one of the following conditions holds.*
(i) There exists a vertex $v \in V - D'$ such that $N_e(v) \cap D' = \{u\}$.
(ii) $N_e(u) \cap D' = \phi$.

Proof. Suppose that d' is an inverse equitable dominating set of G and let conditions (i) and (ii) not hold. Then for some vertex $u \in D'$ there exists $v \in N_e(u) \cap D'$. Therefore $D' - \{u\}$ is an equitable dominating set of G, a contradiction with the minimality of D'.

Conversely, Let for every $u \in D^{'}$ one of the conditions (i) or (ii) holds.Suppose that $D^{'}$ is not minimal. Then there exists $u \in D^{'}$ such that $D^{'} - \{u\}$ is an equitable dominating set of G. That means there exists $v \in D^{'} - \{u\}$ which is equitable adjacent to u. Hence (ii) does not satisfy.

Theorem 2.9. *For any graph G with no isolated vertices,*

$$\gamma(G) \leq \min\{\gamma^{-1}(G), \gamma_e(G), \gamma_e^{-1}(G)\}.$$

Proof. Since every inverse equitable dominating set is inverse dominating sets of G and every inverse dominating set is dominating set, similarly every equitable dominating set is dominating set. Hence $\gamma(G) \leq \min\{\gamma^{-1}(G), \gamma_e(G), \gamma_e^{-1}(G)\}$. Obviously we ask the natural question when $\gamma_e(G) = \gamma_e^{-1}(G)$.

In the following result gives a sufficient condition for a graph G with no equitable isolated vertices to have $\gamma_e(G) = \gamma_e^{-1}(G)$.

Theorem 2.10. *Let G be a graph without equitable isolated vertices and let D be a minimum equitable dominating set. If for every $v \in D$ the induced subgraph $\langle N_e[v] \rangle$ is a complete graph of order at least 2, then* $\gamma_e(G) = \gamma_e^{-1}(G)$.

Proof. Let $D = \{v_1, v_2, ..., v_{\gamma_e(G)}\}$ be a minimum equitable dominating set of G and $u_1, u_2, ..., u_{\gamma_e(G)}\}$ be the vertices which they are equitable adjacent to $v_1, v_2, ..., v_{\gamma_e(G)}\}$ respectively. Since for each vertex $v_i \in D$ the graph $\langle N_e[v] \rangle$ is complete. Then $N_e[v_i] \subset N_e[u_i]$. Hence $V(G) = \bigcup_{i=1}^{\gamma_e(G)} N_e[v_i] \subset \bigcup_{i=1}^{\gamma_e(G)} N_e[u_i] = V(G)$. Thus $\{u_1, u_2, ..., u_{\gamma_e(G)}\} = D^{'}$ is an inverse equitable dominating set of G that means $\gamma^{-1}(G) \leq |D^{'}| = |D| = \gamma_e(G)$. Also by Observation 2.6, we have $\gamma_e(G) \leq \gamma_e^{-1}(G)$. Hence $\gamma_e(G) = \gamma_e^{-1}(G)$.

Theorem 2.11. *Let G be a graph with p vertices and Let τ denote the family of minimum equitable dominating set of G. If for any $D \in \tau$ we have $V - D$ is independent set, then $\gamma_e(G) + \gamma_e^{-1}(G) = p$.*

Proof. Since for any $D \in \tau$ we have $V - D$ is independent set, then $V - D$ is minimum inverse equitable dominating set. Hence $\gamma_e(G) + \gamma_e^{-1}(G) = p$.

Theorem 2.12. *Let G be graph with (p, q) graph and has no equitable isolated vertices and $\gamma_e(G) = \gamma_e^{-1}(G)$. Then $\frac{2p - q}{3} \leq \gamma_e^{-1}(G)$.*

Proof. Let D and $D^{'}$ be the minimum equitable dominating set and the corresponding inverse equitable dominating set of G respectively. Let $A = \{V - (D \cup D^{'}\}$ obviously $|A| = p - 2\gamma_e^{-1}(G)$. Now each vertex in A has at least one edge to D and at least one edge to $D^{'}$. Therefore

$$q \geq 2(p - 2\gamma_e^{-1}(G)) + \gamma_e^{-1}(G).$$

Hence

$$\frac{2p - q}{3} \leq \gamma_e^{-1}(G).$$

Corollary 2.13. *For any tree T_p without any equitable isolated vertices,* $\frac{p+1}{3} \leq \gamma_e^{-1}(T_p)$.

3 INVERSE EQUITABLE EDGE DOMINATION NUMBER

Anwar Alwardi and N. D. Soner introduce the Edge Equitable Domination in graphs [1]. Let $G = (V, E)$ be a graph.

for any edge $f \in E$ The degree of $f = uv$ in G is defined by $deg(f) = deg(u) + deg(v) - 2$. A set $S \subseteq E$ of edges is equitable edge dominating set of G if every edge f not in S is adjacent to at least one edge $f^{'} \in S$ such that $|deg(f) - deg(f^{'})| \leq 1$.

The minimum cardinality of such equitable edge dominating set is denoted by $\gamma_e^{'}(G)$ and is called equitable edge domination number of G. S is minimal if for any edge $f \in S$, $S - \{f\}$ is not an equitable edge dominating set of G. A subset S of E is called an equitable edge independent set, if for any $f \in S$, $f \notin N_e(g)$, for all $g \in S - \{f\}$. If an edge $f \in E$ be such that $|deg(f) - deg(g)| \geq 2$ for all $g \in N(f)$ then f is in any equitable dominating set. Such edges are called equitable isolates. The equitable neighbourhood of f denoted by $N_e(f)$ is defined as $N_e(f) = \{g \in N(f), |deg(f) - deg(g)| \leq 1\}$. The cardinality of $N_e(f)$ is called the equitable degree of f and denoted by $deg_e(f)$. The maximum and minimum equitable degree of edge in G are denoted respectively by $\Delta_e^{'}(G)$ and $\delta_e^{'}(G)$. That is $\Delta_e^{'}(G) = max_{f \in E(G)} |N_e(f)|$, $\delta_e^{'}(G) = min_{f \in E(G)} |N_e(f)|$. The equitable degree of an edge f in a graph G denoted by $deg_e(f)$ is equal to the number of edges which is equitable adjacent with f. the minimum equitable edge dominating set is denoted by $\gamma_e^{'}$-set. In this paper if f and g any two edges in $E(G)$ we say that f and g are equitable adjacent if f and g are adjacent and $|deg(f) - deg(g)| \leq 1$ where $deg(f), deg(g)$ is the degree of the edges f and g respectively. The degree of the edge $f = uv$, $deg(f) = deg(v) + deg(u) - 2$.

Definition 3.1. L*et* F *be minimum equitable edge dominating set of a graph* $G = (V, E)$. *If* $E - F$ *contains an equitable edge dominating set* $F^{'}$*, then* $F^{'}$ *called an inverse edge equitable dominating set of* G *with respect to* F. *The minimum number of edges in an inverse edge equitable dominating set of* G *is called the inverse equitable edge domination number and denoted by* $(\gamma_e^{'})^{-1}(G)$.

Example. Let G be a graph as in Figure 3, then the minimum equitable edge dominating set is $\{e_2, e_4, e_6\}$ and the minimum equitable edge dominating sets are $\{e_1, e_3, e_5, e_7\}$ and $\{e_1, e_7, e_8, e_9\}$. Hence $\gamma_e^{'}(G) = 3$ and $(\gamma_e^{'})^{-1}(G) = 4$.

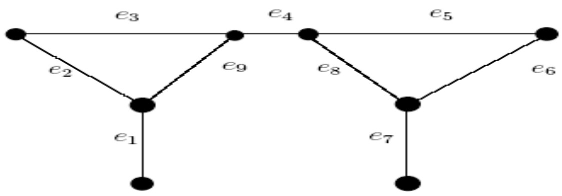

Figure: 3

Obviously the inverse equitable edge dominating set exist if G has no equitable isolated edge.

Theorem 3.2. *A graph* $G = (V, E)$ *has an inverse equitable edge dominating set if and only if* G *has no equitable isolated edge.*

Proof. If $G = (V, E)$ has no equitable isolated edge and F is an equitable edge dominating set, then $E - F$ is an inverse equitable edge dominating set.

Conversely, Let G has an equitable edge dominating set F and an inverse equitable edge dominating set $F^{'}$. Suppose G has an equitable isolated edge f, then f must belong to F and $F^{'}$, a contradiction.

Proposition 3.3. *For any graph* G *without isolated edges,* $\gamma_e^{'}(G) \leq (\gamma_e^{'})^{-1}(G)$, *where* $\gamma_e^{'}(G)$ *is the equitable edge domination number of* G.

Proof. Since each inverse equitable edge dominating set of a graph G is equitable edge dominating set then the proof is straightforward.

Theorem 3.4 *Let F be a minimum equitable edge dominating set of G. If for each edge $f \in F$, the induced subgraph $N_e(f)$ is star, then*

$$\gamma_e^{'}(G) = (\gamma_e^{'})^{-1}(G).$$

Proof. Let F be a minimum equitable edge dominating set of G. Since for each edge $f \in F$, the induced subgraph $N_e(f)$ is star, then $F^{'} = \{f^{'} : f^{'}$ is equitable adjacent to $f \in F\}$ is a minimum inverse equitable edge dominating set. Thus

$$(\gamma_e^{'})^{-1}(G) = |F^{'}| = |F| = \gamma_e^{'}(G).$$

In the following Proposition we list the inverse equitable edge domination number for some standard graph. the proof of this proposition is straightforward so we omitted the proof.

Proposition 3.5. *For any complete graph K_p, Path P_p, Cycle graph C_p and complete bipartite graph $K_{m,n}$, we have:*

(i) $(\gamma_e^{'})^{-1}(K_p) = \lfloor \frac{p}{2} \rfloor$, $p \geq 3$.

(ii) $(\gamma_e^{'})^{-1}(P_p) = \lceil \frac{p}{3} \rceil$, $p \geq 3$.

(iii) $(\gamma_e^{'})^{-1}(C_p) = \lceil \frac{p}{3} \rceil$, $p \geq 3$.

(iv) $(\gamma_e^{'})^{-1}(K_{m,n}) = \min\{m,n\}$, $m,n \geq 2$.

Theorem 3.6 [2] *For any (p,q) graph G, $\lceil \frac{q}{\Delta_e^{'}(G)+1} \rceil \leq \gamma_e^{'}(G) \leq q - \beta_e^{'} + q_0$, where q_0 is the number of equitable isolated edges.*

Theorem 3.7 *If G is a graph without equitable isolated edges and $p \geq 3$, then*

$$\gamma_e^{'}(G) + (\gamma_e^{'})^{-1}(G) \leq q.$$

Proof. let $G = (V, E)$ be any graph without any equitable isolated edges and let F and $F^{'}$ be the minimum equitable and inverse equitable dominating sets of G. Then $F \cup F^{'} \subseteq E(G)$, Hence $\gamma_e^{'}(G) + (\gamma_e^{'})^{-1}(G) \leq q$.

Theorem 3.8. *If G is a graph without equitable isolated edges and $p \geq 3$, then*

$$(\gamma_e^{'})^{-1}(G) \leq \lceil \frac{q\Delta_e^{'}(G)}{\Delta_e^{'}(G)+1} \rceil.$$

Further the equality holds if $G = P_3$ or P_4.

Proof. By Theorem 3.6, we have $\lceil \frac{q}{\Delta_e^{'}(G)+1} \rceil \leq \gamma_e^{'}(G)$ and by Theorem 3.7, we have $\gamma_e^{'}(G) + (\gamma_e^{'})^{-1}(G) \leq q$, therefore

$$(\gamma_e^{'})^{-1}(G) \leq q - \gamma_e^{'}(G) \leq q - \lceil \frac{q}{\Delta_e^{'}(G)+1} \rceil = \lceil \frac{q\Delta_e^{'}(G)}{\Delta_e^{'}(G)+1} \rceil$$

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