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A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS ON A PROBABILISTIC METRIC SPACE UNDER DNR COMMUTATIVITY CONDITION USING IMPLICIT RELATION

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ABSTRACT

The aim of present paper is to obtain a common fixed point theorem for four self mappings on a probabilistic metric space by using DNR-commutativity in probabilistic metric spaces satisfying implicit relations.

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Key Words: *probabilistic metric space, reciprocally continuous, DNR-commuting, implicit relation.*

1. INTRODUCTION

In 1942, K. Menger [5] introduced the notion of a probabilistic metric space (briefly PM-space) as a generalization of metric space. The development of the theory of probabilistic metric spaces is due to Schweizer and Sklar [11]. Sehgal [12] initiated study of fixed point theory in PM space contraction mapping theorems in PM-spaces.

Generalization of the notion of commutativity of mappings has been extended to PM-spaces by various authors. Singh and Pant [15] extended the notion of weak commutativity (introduced by Sessa [13] in metric spaces). Mishra [7] extended the notion of compatibility (introduced by Jungck [2] in metric spaces). Ciric and Milovanovic –Arandjelovic [1] extended the notion of point wise R-weak commutativity (introduced by Pant [8] in metric spaces). In 2007, Kohli, Vasista [3] extended the notion of R-weak commutativity and its variants to probabilistic metric spaces.

In 2012, Shikha Chaudhari [14] established the existence of a common fixed point for six mappings in PM-spaces satisfying implicit relation and variants of R-weak commutativity.

Recently K.P.R. Sastry. et.al [10] introduced the notion DNR-functions and DNR -commutativity as a generalization of R-weak commutativity.

In this paper we use DNR-commutativity instead of R-weak commutativity in [14] for four mappings and latter extended for six mappings.

In this paper we establish a common fixed point theorem for four self maps on a Menger space, satisfying DNR-commutativity property.

This result is also extended to six self maps.

2. PRELIMINARIES

Throughout the paper, $\mathbb{R}$ stands for the real line and $\mathbb{R}^+$ stands for the set of non negative real numbers. We begin with some definitions.

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Definition 2.1: [11] A mapping $F: \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We shall denote by $\mathfrak{D}$, the class of all distribution functions.
The Heaviside function H is a distribution function defined by $H(t) = \begin{cases} 0 \; if \; t \le 0 \\ 1 \; if \; t > 0 \end{cases}$

Definition 2.2: [11] Let X be a non empty set and let $\mathfrak{D}$ denote the set of all distribution functions. An ordered pair (X, F) is called a probabilistic metric space if F is a mapping from $X \times X \to \mathfrak{D}$ satisfying the following conditions.

(1) $F_{u,v}(t) = H(t)$ if and only if $x = y$,
(2) $F_{x,y}(0) = 0$
(3) $F_{x,y}(t) = F_{y,x}(t)$
(4) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

Definition 2.3: [11] A t-norm is a function $t: [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions.

(1) $t(a,1) = a \quad \forall\, a \in [0,1]$
(2) $t(a,b) = t(b,a)$
(3) $t(c,d) \ge t(a,b) \; for \; c \ge a \; and \; d \ge b$
(4) $t(t(a,b),c) = t(a,t(b,c)) \;\; \forall\, a,b,c \in [0,1]$

Examples of t-norms are $t(a,b) = min\{a,b\}$, $t(a,b) = ab$ and $t(a,b) = min\{a+b-1, 0\}$.

Definition 2.4: [11] A Menger probabilistic metric space (X, F, t) is an ordered triad, where t is a t-norm and (X, F) is a probabilistic metric space satisfying:

$$F_{x,z}(t+s) \ge t\left(F_{x,y}(t), F_{y,z}(s)\right) \;\; \forall\; t, s \ge 0 \; and \; x, y, z \in X.$$

Definition 2.5: [11] A sequence $\{x_n\}$ in (X, F, t) is said to converge to $x \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \ge N(\varepsilon, \lambda)$.

Definition 2.6: [11] A sequence $\{x_n\}$ in (X, F, t) is said to be a Cauchy sequence if for $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_m, x_n}(\varepsilon) > 1 - \lambda$ for all $m, n > N(\varepsilon, \lambda)$.

Definition 2.7: [11] A Menger space (X, F, t) with continuous t-norm, is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.8: [14] Two self mappings f and g of a probabilistic metric space (X, F) are said to be weakly commuting if $F_{fgx, gfx}(t) \ge F_{fx, gx}(t)$ for each $x \in X$ and $t > 0$

Definition 2.9: [7] Two self mappings f and g of a probabilistic metric space (X, F) will be compatible if and only if $\lim_{n\to\infty} F_{fgx_n, gfx_n}(t) = 1 \; \forall\, t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \; for \; some \; z \in X$.

Definition 2.10: [1] Two self mappings f and g of a probabilistic metric space (X, F) are said to be point wise R-weakly commuting if given $x \in X$, there exists $R > 0$ such that $F_{fgx, gfx}(t) \ge F_{fx, gx}\left(\frac{t}{R}\right)$ for $t \ge 0$.

Definition 2.11: [14] Two self mappings f and g of a probabilistic metric space (X, F) are said to be reciprocally continuous if $fgx_n \to fz$ and $gfx_n \to gz$ whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to z \; for \; some \; z \in X$.

Definition 2.12: [3] Two self mappings f and g of a probabilistic metric space (X, F) are said to be

(I) R-weakly commuting of type (i) if there exists a positive real number R such that

$$F_{ffx, gfx}(t) \ge F_{fx, gx}\left(\frac{t}{R}\right) \text{ for each } \; x \in X \; and \; t \ge 0.$$

(II) R-weakly commuting of type (ii) if there exists a positive real number R such that

$$F_{fgx, ggx}(t) \ge F_{fx, gx}\left(\frac{t}{R}\right) \text{ for each } \; x \in X \; and \; t \ge 0.$$

(III) R-weakly commuting of type (iii) if there exists a positive real number R such that

$$F_{ffx,ggx}(t) \geq F_{fx,gx}\left(\frac{t}{R}\right) \text{ for each } \quad x \in X \text{ and } t \geq 0.$$

Lemma 2.13: [4] Let $\{u_n\}$ be a sequence in a Menger space(X, F, t). If there exists a constant $h \in (0,1)$ such that $F_{u_n,u_{n+1}}(ht) \geq F_{u_{n-1},u_n}(t),\ n = 1,2,3,...$,then $\{u_n\}$ is a Cauchy sequence in X.

3. IMPLICIT RELATIONS

In [6] Mihet established a fixed point theorem concerning probabilistic contraction satisfying an implicit relation. In [9] Popa used the family F_4 of implicit real functions to find the fixed point of two pairs of semi compatible mappings in a d-compatible topological space. Here F_4 denotes the family of all real continuous functions $f: (\mathbb{R})^4 \to \mathbb{R}$ satisfying the following properties.

(F_K) there exists $k \geq 1$ such that for every $u \geq 0, v \geq 0$ with $f(u,v,u,v) \geq 0$ (or) $f(u,v,v,u) \geq 0$ we have $u \geq kv$

(F_u) $f(u,v,0,0) < 0$ $for\ all\ u > 0.$

We denote by Φ the class of all real valued continuous functions $\varphi: (\mathbb{R}^+)^4 \to \mathbb{R}$, non deceasing in first argument and satisfying

(i) for all $u, v \geq 0$, $\varphi(u,v,u,v) \geq 0$ (or) $\varphi(u,v,v,u) \geq 0 \Rightarrow u \geq v$ (3.1)

(ii) $\varphi(u,u,1,1) \geq 0$ $for\ all\ u \geq 1$ (3.2)

4. MAIN RESULTS

Kohli, Vashistha and Kumar [4] proved the following lemma for six mappings.

Lemma 4.1 (Lemma4.2, Kohli , Vashistha and Kumar [4]): Let (X, F, t) be a complete Menger space where T denotes a continuous t-norm. Further, let (p, hk) and (q, fg) be pointwise R-weakly commuting pairs of self-mappings of X satisfying

$$p(X) \subset fg(X), q(X) \subset hk(X) \tag{4.1.1}$$

$$\varphi(F_{px,qy}(\alpha t), F_{hkx,fgy}(t), F_{px,hkx}(t), F_{qy,fgy}(\alpha t) \geq 0 \tag{4.1.2}$$

$$\varphi(F_{px,qy}(\alpha t), F_{hkx,fgy}(t), F_{px,hkx}(\alpha t), F_{qy,fgy}(t) \geq 0 \tag{4.1.3}$$

for all $x, y \in X, t > 0, \alpha \in (0,1)$ and for some $\varphi \in \Phi$.

Then the continuity of one of the mappings in the compatible pairs (p, hk) or (q, fg) on (X, F, T) implies their reciprocal continuity.

Shikha Chaudhari [14] proved the lemma by assuming (p, hk) and (q, fg) to be R-weakly commuting mappings of type (i), type (ii) and type (iii) respectively.

Now we prove an analogue of the above Lemma 4.1 for four maps.

Lemma 4.2: Let (X, F, t) be a complete Menger space where t denotes a continuous t-norm. Further, let A, S, B and T be self mappings on X satisfying

$$A(X) \subset T(X), B(X) \subset S(X) \tag{4.2.1}$$

$$\varphi\left(F_{Ax,By}(ht), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(ht)\right) \geq 0 \tag{4.2.2}$$

$$\varphi\left(F_{Ax,By}(ht), F_{Sx,Ty}(t), F_{Ax,Sx}(ht), F_{By,Ty}(t)\right) \geq 0 \tag{4.2.3}$$

for all $x, y \in X, t > 0, h \in (0,1)$ and for some $\varphi \in \Phi$.

Moreover A and S are commuting and B and T are commuting. If

(i) S is continuous then (A, S) is reciprocally continuous.

(ii) T is continuous then (B, T) is reciprocally continuous.

Proof: We prove (i). The proof of (ii) is similar. Suppose $A\ and\ S$ commute, so that $A\ and\ S$ are compatible. Suppose S is continuous.

We shall show that $A\ and\ S$ are reciprocally continuous.

Let $\{x_n\}$ be a sequence in X such that $Ax_n \to z$ and $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Since S is continuous $SAx_n \to Sz$, $SSx_n \to Sz$.

We show that $ASx_n \to Az$.

In view of compatibility of (A, S), we have $F_{ASx_n,SAx_n}(t) \to 1$.

i.e. $F_{ASx_n,Sz}(t) \to 1$
i.e. $ASx_n \to Sz$ as $n \to \infty$.

In view of (4.2.1) for each n, there exists $y_n \in X$ such that $ASx_n = Ty_n$.

So $SSx_n \to Sz$, $SAx_n \to Sz$, $ASx_n \to Sz$ and $Ty_n \to Sz$ as $n \to \infty$. (4.2.4)

Next we claim that $By_n \to Sz$ as $n \to \infty$.

By putting $x = Sx_n$ and $y = y_n$ in (4.2.2), we get

$$\varphi\left(F_{ASx_n,By_n}(ht), F_{SSx_n,Ty_n}(t), F_{ASx_n,SSx_n}(t), F_{By_n,Ty_n}(ht)\right) \geq 0$$

Since φ is continuous, by (4.2.4), we have
$$\varphi\left(F_{Sz,By_n}(ht), 1, 1, F_{By_n,Sz}(ht)\right) \geq 0$$

i.e. $F_{Sz,By_n}(ht) \geq 1$ (from (3.1))

$\therefore\ F_{Sz,By_n}(ht) = 1$

i.e. $By_n \to Sz$ as $n \to \infty$.

Again putting $x = z$ and $y = y_n$ in (4.2.3), we get
$$\varphi\left(F_{Az,By_n}(ht), F_{Sz,Ty_n}(t), F_{Az,Sz}(ht), F_{By_n,Ty_n}(t)\right) \geq 0$$

Letting $n \to \infty$, we get

$$\varphi\left(F_{Az,Sz}(ht), F_{Sz,Sz}(t), F_{Az,Sz}(ht), F_{Sz,Sz}(t)\right) \geq 0$$

i.e $\varphi\left(F_{Az,Sz}(ht), 1, F_{Az,Sz}(ht), 1\right) \geq 0$

By (3.1), we get $F_{Az,Sz}(ht) \geq 1$

$\therefore Az = Sz$

Hence $ASx_n \to Az$.

This completes the proof of the lemma.

Now we introduce the notion of a DNR-function and DNR-commuting property.

Definition 4.3: [10] A function $\psi: X \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a DNR function if

$\psi(x, t) > 0\ for\ all\ x \in X\ and\ t > 0.$

Ψ denotes the class of all DNR functions ψ.

Example 4.4: [10] Let $X = \{2,3,4,\ldots\}$ with the metric $d(x,y) = |x-y|$ and define

$$F_{x,y}(t) = \begin{cases} 0 \ \ if \ \ t \le x \\ 1 \ \ if \ \ t > y \\ \frac{t-x}{y-x} \ \ if \ \ x < t \le y \end{cases} \quad \text{for } x < y.$$

Clearly (X,F) is a PM-space. Define $\psi(x,t) = \begin{cases} x \ \ if \ \ t \le x \\ \frac{t-1}{x} \ \ if \ \ t > x \end{cases}$ for $x \in [2,\infty)$

Then ψ is a DNR function.

Definition 4.5: [10] Suppose A and S are self maps on a PM-space (X,F). We say that the pair (A,S) is DNR-commutating if $z \in X$ and $t > 0 \Rightarrow$ there exists $\psi \in \Psi$ such that $F_{ASz,SAz}(t) \ge F_{Az,Sz}(\psi(z,t))$.

Note 1: If $A \ and \ S$ are point wise R-weakly commuting self maps on a PM- space X**,** then $A \ and \ S$ are DNR-commuting.

Note 2: If $A \ and \ S$ are DNR-commuting self maps on a PM-space X, then $A \ and \ S$ are compatible.

Theorem 4.6: Let (X,F,t) be a complete Menger space where t denotes a continuous t-norm. Further, let $(A,S) and \ (B,T)$ be DNR-commuting pairs of self mappings on X satisfying

$$A(X) \subset T(X), B(X) \subset S(X) \tag{4.6.1}$$

$$\varphi\left(F_{Ax,By}(ht), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(ht)\right) \ge 0 \tag{4.6.2}$$

$$\varphi\left(F_{Ax,By}(ht), F_{Sx,Ty}(t), F_{Ax,Sx}(ht), F_{By,Ty}(t)\right) \ge 0 \tag{4.6.3}$$

for all $x,y \in X, t > 0, h \in (0,1)$ and for some $\varphi \in \Phi$.

Moreover S commutes with A and T commutes with B. Further suppose that S and T are continuous. Then $A, B, S \ and \ T$ have a unique common fixed point in X.

Proof: Let $u_0 \in X$. By (4.6.1), we define the sequences $\{u_n\}$ and $\{v_n\}$ in X such that for $n = 0,1,2,\ldots$

$v_{2n+1} = Au_{2n} = Tu_{2n+1}$,

$v_{2n+2} = Bu_{2n+1} = Su_{2n+2}$

Now by putting $x = u_{2n}, y = u_{2n+1}$ in (4.6.2), we get

$$\varphi\left(F_{Au_{2n},Bu_{2n+1}}(ht), F_{Su_{2n},Tu_{2n+1}}(t), F_{Au_{2n},Su_{2n}}(t), F_{Bu_{2n+1},Tu_{2n+1}}(ht)\right) \ge 0$$

$$\Rightarrow \varphi\left(F_{v_{2n+1},v_{2n+2}}(ht), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n+1},v_{2n}}(t), F_{v_{2n+2},v_{2n+1}}(ht)\right) \ge 0$$

Using (3.1), we get

$$F_{v_{2n+1},v_{2n+2}}(ht) \ge F_{v_{2n},v_{2n+1}}(t)$$

Now by putting $x = u_{2n+2}, y = u_{2n+1}$ in (4.6.2), we get

$$\varphi\left(F_{Au_{2n+2},Bu_{2n+1}}(ht), F_{Su_{2n+2},Tu_{2n+i}}(t), F_{Au_{2n+2},Su_{2n+2}}(t), F_{Bu_{2n+1},Tu_{2n+1}}(ht)\right) \ge 0$$

$$\Rightarrow \varphi\left(F_{v_{2n+3},v_{2n+2}}(ht), F_{v_{2n+2},v_{2n+1}}(t), F_{v_{2n+3},v_{2n+2}}(t), F_{v_{2n+2},v_{2n+1}}(ht)\right) \ge 0$$

Using (3.1), we get

$$F_{v_{2n+3},v_{2n+2}}(ht) \ge F_{v_{2n+2},v_{2n+1}}(t)$$

Thus for any n and t, we have $F_{v_n,v_{n+1}}(ht) \geq F_{v_{n-1},v_n}(t)$.

Hence by Lemma 2.13, $\{v_n\}$ is a Cauchy sequence in X.

Since X is complete $\{v_n\}$ converges to z.

$\therefore$ Its subsequences $\{Au_{2n}\}, \{Bu_{2n+1}\}, \{Su_{2n}\}$ and $\{Tu_{2n+1}\}$ also converge to z.

Now, suppose that (A, S) is a compatible pair and S is continuous. Then by Lemma 4.2, A and S are reciprocally continuous.

Then $ASu_{2n} \to Az$ and $SAu_{2n} \to Sz$.

Compatibility of A and S gives $F_{ASu_{2n},SAu_{2n}}(t) \to 1$.

i.e. $F_{Az,Sz}(t) \to 1$ as $n \to \infty$.

Hence $Az = Sz$.

Since $A(X) \subset S(X)$, there exists a point u in X such that $Az = Tu$.

Now by putting $x = z, y = u$ in (4.6.2), we get

$$\varphi\left(F_{Az,Bu}(ht), F_{Sz,Tu}(t), F_{Az,Sz}(t), F_{Bu,Tu}(ht)\right) \geq 0$$

i.e. $\varphi\left(F_{Az,Bu}(ht), 1, 1, F_{Bu,Az}(ht)\right) \geq 0$

Using (3.1), we get $F_{Az,Bu}(ht) \geq 1$ for all $t \geq 0$

$\Rightarrow F_{Az,Bu}(ht) = 1$

Hence $Az = Bu$.

Thus $Az = Sz = Bu = Tu$.

Since A and S are DNR-commuting, to the pair (z, t) corresponds a $\psi \in \Psi$ such that

$$F_{ASz,SAz}(t) \geq F_{Az,Sz}(\psi(z,t))$$
$$= 1$$

Hence $ASz = SAz$ and $SAz = SSz = AAz = ASz$.

Since B and T are DNR-commuting, we have

$BBu = BTu = TBu = TTu$.

Again by putting $x = Az, y = u$ in (4.6.2), we get

$$\varphi\left(F_{AAz,Bu}(ht), F_{SAz,Tu}(t), F_{AAz,SAz}(t), F_{Bu,Tu}(ht)\right) \geq 0$$

$\Rightarrow \varphi\left(F_{AAz,Az}(ht), F_{AAz,Az}(t), 1, 1\right) \geq 0$

Therefore $F_{AAz,Az}(ht) \geq 1$ $for\ all$ $t > 0$, using (3.1)

$\therefore$ $F_{AAz,Az}(ht) = 1$.

$\Rightarrow AAz = Az$ and $Az = AAz = SAz$.

$\therefore$ Az is a common fixed point of A and S.

By putting $x = z, y = Bu$ in (4.6.3), we get

$$\varphi\left(F_{Az,BBu}(ht), F_{Sz,TBu}(t), F_{Az,Sz}(ht), F_{BBu,TBu}(t)\right) \geq 0$$

$$\Rightarrow \varphi\left(F_{Az,BAz}(ht), F_{Az,TAz}(t), 1,1\right) \geq 0$$

Since φ is non decreasing, using (3.1), we get

$F_{Az,BAz}(t) \geq 1$. ($\because BAz = BTu = TBu = TAz$)

$\therefore Az = BAz = TAz$.

Hence Az is a common fixed point of A, B, S and T.

Clearly, Az is the unique common fixed point of A, B, S and T.

Note: The theorem is valid if one of the mappings in the compatible pairs (A, S) and (B, T) is continuous instead of assuming that S and T are continuous. The proof is similar.

The following theorem is an extension to six mappings

Theorem 4.7: Let (X, F, t) be a complete Menger space, where t denotes a continuous t-norm. Suppose A, B, S, T, H and R are self maps on X such that (A, SH) and (B, TR) are DNR-commuting pairs of self mappings on X satisfying

$A(X) \subset TR(X), B(X) \subset SH(X)$ (4.7.1)

$$\varphi\left(F_{Ax,By}(\alpha t), F_{SHx,TRy}(t), F_{Ax,SHx}(t), F_{By,TRy}(\alpha t)\right) \geq 0 \quad (4.7.2)$$

$$\varphi\left(F_{Ax,By}(\alpha t), F_{SHx,TRy}(t), F_{Ax,SHx}(\alpha t), F_{By,TRy}(t)\right) \geq 0 \quad (4.7.3)$$

for all $x, y \in X, t > 0, \alpha \in (0,1)$ and for some $\varphi \in \Phi$.

Moreover suppose H commutes with A and S and R commutes with B and T. Suppose that SH and TR are continuous. Then A, B, S, T, R and H have a unique common fixed point in X.

Proof: Write $SH = P$ and $TR = Q$.

By hypothesis, P and Q are continuous.

Thus by Theorem 4.6, A, B, P and Q have a unique common fixed point z in X.

i.e. $Az = Bz = Pz = Qz = z$.

Take $x = Hz,\ y = z$ in (4.7.2). We get

$$\varphi\left(F_{AHz,Bz}(\alpha t), F_{SHHz,TRz}(t), F_{AHz,SHHz}(t), F_{Bz,TRz}(\alpha t)\right) \geq 0$$

$$\varphi\left(F_{HAz,z}(\alpha t), F_{HSHz,Qz}(t), F_{HAz,HSHz}(t), F_{z,Qz}(\alpha t)\right) \geq 0$$

$$\varphi\left(F_{Hz,z}(\alpha t), F_{Hz,z}(t), F_{Hz,Hz}(t), F_{z,z}(\alpha t)\right) \geq 0$$

Hence $\varphi\left(F_{Hz,z}(\alpha t), F_{Hz,z}(t), 1,1\right) \geq 0$

$\Rightarrow F_{Hz,z}(t) \geq 1$ $for\ every$ $t > 0$ ($\because \varphi$ is non decreasing in its first co ordinate)

$\Rightarrow Hz = z$

$\Rightarrow z$ is a fixed point of H.

Now $z = Pz = SHz = Sz$.

$\therefore$ z is also a fixed point of S.

Now take $x = z$ and $y = Tz$ in (4.7.2). We get

$\varphi\left(F_{Az,BTz}(\alpha t), F_{SHz,TRTz}(t), F_{Az,SHz}(t), F_{BTz,TRTz}(\alpha t)\right) \geq 0$

$\Rightarrow \varphi\left(F_{z,TBz}(\alpha t), F_{z,TTRz}(t), F_{z,z}(t), F_{TBz,TTRz}(\alpha t)\right) \geq 0$

$\Rightarrow \varphi\left(F_{z,Tz}(\alpha t), F_{z,Tz}(t), F_{z,z}(t), F_{Tz,Tz}(\alpha t)\right) \geq 0$

$\Rightarrow \varphi\left(F_{z,Tz}(\alpha t), F_{z,Tz}(t), 1, 1\right) \geq 0$

$\Rightarrow F_{z,Tz}(t) \geq 1$ for every $t > 0$ ($\because \varphi$ is non decreasing in its first co ordinate)

$\Rightarrow Tz = z$.

$\therefore$ z is also a fixed point of T

Now $z = Qz = TRz = RTz = Rz$.

$\therefore$ z is also a fixed point of R

Hence z is a common fixed point of $A, B, S, T, R \text{ and } H$.

Suppose x is also a fixed point of $A, B, S, T, R \text{ and } H$.

Then $Pz = SHz = S(Hz) = Sz = z$

and $Qz = TRz = T(Rz) = Tz = z$

Similarly $Qx = Px = x$. Thus x and z are common fixed points of $A, B, P \text{ and } Q$.

Hence by Theorem 4.6, $z = x$.

Thus $A, B, S, T, R \text{ and } H$ have unique fixed point.

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