



A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS ON A  
PROBABILISTIC METRIC SPACE UNDER DNR COMMUTATIVITY CONDITION  
USING IMPLICIT RELATION

K. P. R. Sastry<sup>1</sup>, G. A. Naidu<sup>2</sup>, D. Narayana Rao<sup>3\*</sup> and S. S. A. Sastri<sup>4</sup>

<sup>1</sup>8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam- 530017, India

<sup>2,3</sup>Department of Mathematics, Andhra University, Visakhapatnam-530 003, India

<sup>4</sup>Department of Mathematics, GVP College of Engineering, Madhurawada,  
Visakhapatnam- 530048, India

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ABSTRACT

The aim of present paper is to obtain a common fixed point theorem for four self mappings on a probabilistic metric space by using DNR-commutativity in probabilistic metric spaces satisfying implicit relations.

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Key Words: probabilistic metric space, reciprocally continuous, DNR-commuting, implicit relation.

1. INTRODUCTION

In 1942, K. Menger [5] introduced the notion of a probabilistic metric space (briefly PM-space) as a generalization of metric space. The development of the theory of probabilistic metric spaces is due to Schweizer and Sklar [11]. Sehgal [12] initiated study of fixed point theory in PM space contraction mapping theorems in PM-spaces.

Generalization of the notion of commutativity of mappings has been extended to PM-spaces by various authors. Singh and Pant [15] extended the notion of weak commutativity (introduced by Sessa [13] in metric spaces). Mishra [7] extended the notion of compatibility (introduced by Jungck [2] in metric spaces). Ciric and Milovanovic –Arandjelovic [1] extended the notion of point wise R-weak commutativity (introduced by Pant [8] in metric spaces). In 2007, Kohli, Vasista [3] extended the notion of R-weak commutativity and its variants to probabilistic metric spaces.

In 2012, Shikha Chaudhari [14] established the existence of a common fixed point for six mappings in PM-spaces satisfying implicit relation and variants of R-weak commutativity.

Recently K.P.R. Sastry. et.al [10] introduced the notion DNR-functions and DNR -commutativity as a generalization of R-weak commutativity.

In this paper we use DNR-commutativity instead of R-weak commutativity in [14] for four mappings and latter extended for six mappings.

In this paper we establish a common fixed point theorem for four self maps on a Menger space, satisfying DNR-commutativity property.

This result is also extended to six self maps.

2. PRELIMINARIES

Throughout the paper,  $\mathbb{R}$  stands for the real line and  $\mathbb{R}^+$  stands for the set of non negative real numbers. We begin with some definitions.

\*Corresponding author: D. Narayana Rao<sup>3\*</sup>,

<sup>3</sup>Department of Mathematics, Andhra University, Visakhapatnam-530 003, India

**Definition 2.1:** [11] A mapping  $F: \mathbb{R} \rightarrow \mathbb{R}^+$  is called a distribution function if it is non-decreasing and left continuous with  $\inf_{t \in \mathbb{R}} F(t) = 0$  and  $\sup_{t \in \mathbb{R}} F(t) = 1$ .

We shall denote by  $\mathfrak{D}$ , the class of all distribution functions.

The Heaviside function  $H$  is a distribution function defined by  $H(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$

**Definition 2.2:** [11] Let  $X$  be a non empty set and let  $\mathfrak{D}$  denote the set of all distribution functions. An ordered pair  $(X, F)$  is called a probabilistic metric space if  $F$  is a mapping from  $X \times X \rightarrow \mathfrak{D}$  satisfying the following conditions.

- (1)  $F_{u,v}(t) = H(t)$  if and only if  $x = y$ ,
- (2)  $F_{x,y}(0) = 0$
- (3)  $F_{x,y}(t) = F_{y,x}(t)$
- (4) If  $F_{x,y}(t) = 1$  and  $F_{y,z}(s) = 1$  then  $F_{x,z}(t+s) = 1$  for all  $x, y, z \in X$  and  $t, s > 0$ .

**Definition 2.3:** [11] A t-norm is a function  $t: [0,1] \times [0,1] \rightarrow [0,1]$  satisfying the following conditions.

- (1)  $t(a, 1) = a \quad \forall a \in [0,1]$
- (2)  $t(a, b) = t(b, a)$
- (3)  $t(c, d) \geq t(a, b)$  for  $c \geq a$  and  $d \geq b$
- (4)  $t(t(a, b), c) = t(a, t(b, c)) \quad \forall a, b, c \in [0,1]$

Examples of t-norms are  $t(a, b) = \min\{a, b\}$ ,  $t(a, b) = ab$  and  $t(a, b) = \min\{a + b - 1, 0\}$ .

**Definition 2.4:** [11] A Menger probabilistic metric space  $(X, F, t)$  is an ordered triad, where  $t$  is a t-norm and  $(X, F)$  is a probabilistic metric space satisfying:

$$F_{x,z}(t+s) \geq t(F_{x,y}(t), F_{y,z}(s)) \quad \forall t, s \geq 0 \text{ and } x, y, z \in X.$$

**Definition 2.5:** [11] A sequence  $\{x_n\}$  in  $(X, F, t)$  is said to converge to  $x \in X$  if for every  $\varepsilon > 0$  and  $\lambda > 0$  there exists a positive integer  $N(\varepsilon, \lambda)$  such that  $F_{x_n,x}(\varepsilon) > 1 - \lambda$  for all  $n \geq N(\varepsilon, \lambda)$ .

**Definition 2.6:** [11] A sequence  $\{x_n\}$  in  $(X, F, t)$  is said to be a Cauchy sequence if for  $\varepsilon > 0$  and  $\lambda > 0$  there exists a positive integer  $N(\varepsilon, \lambda)$  such that  $F_{x_m,x_n}(\varepsilon) > 1 - \lambda$  for all  $m, n > N(\varepsilon, \lambda)$ .

**Definition 2.7:** [11] A Menger space  $(X, F, t)$  with continuous t-norm, is said to be complete if every Cauchy sequence in  $X$  is convergent.

**Definition 2.8:** [14] Two self mappings  $f$  and  $g$  of a probabilistic metric space  $(X, F)$  are said to be weakly commuting if  $F_{fgx, gfx}(t) \geq F_{fx, gxx}(t)$  for each  $x \in X$  and  $t > 0$

**Definition 2.9:** [7] Two self mappings  $f$  and  $g$  of a probabilistic metric space  $(X, F)$  will be compatible if and only if  $\lim_{n \rightarrow \infty} F_{fgx_n, gfx_n}(t) = 1 \quad \forall t > 0$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$  for some  $z \in X$ .

**Definition 2.10:** [1] Two self mappings  $f$  and  $g$  of a probabilistic metric space  $(X, F)$  are said to be point wise R-weakly commuting if given  $x \in X$ , there exists  $R > 0$  such that  $F_{fgx, gfx}(t) \geq F_{fx, gxx}\left(\frac{t}{R}\right)$  for  $t \geq 0$ .

**Definition 2.11:** [14] Two self mappings  $f$  and  $g$  of a probabilistic metric space  $(X, F)$  are said to be reciprocally continuous if  $fgx_n \rightarrow fz$  and  $gfx_n \rightarrow gz$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $fx_n, gx_n \rightarrow z$  for some  $z \in X$ .

**Definition 2.12:** [3] Two self mappings  $f$  and  $g$  of a probabilistic metric space  $(X, F)$  are said to be

- (I) R-weakly commuting of type (i) if there exists a positive real number  $R$  such that

$$F_{ffx, gfx}(t) \geq F_{fx, gxx}\left(\frac{t}{R}\right) \text{ for each } x \in X \text{ and } t \geq 0.$$

- (II) R-weakly commuting of type (ii) if there exists a positive real number  $R$  such that

$$F_{fgx, gxx}(t) \geq F_{fx, gxx}\left(\frac{t}{R}\right) \text{ for each } x \in X \text{ and } t \geq 0.$$

(III) R-weakly commuting of type (iii) if there exists a positive real number  $R$  such that

$$F_{ffx,ggx}(t) \geq F_{fx,gx}\left(\frac{t}{R}\right) \text{ for each } x \in X \text{ and } t \geq 0.$$

**Lemma 2.13:** [4] Let  $\{u_n\}$  be a sequence in a Menger space  $(X, F, t)$ . If there exists a constant  $h \in (0,1)$  such that  $F_{u_n, u_{n+1}}(ht) \geq F_{u_{n-1}, u_n}(t)$ ,  $n = 1, 2, 3, \dots$ , then  $\{u_n\}$  is a Cauchy sequence in  $X$ .

### 3. IMPLICIT RELATIONS

In [6] Mihet established a fixed point theorem concerning probabilistic contraction satisfying an implicit relation. In [9] Popa used the family  $F_4$  of implicit real functions to find the fixed point of two pairs of semi compatible mappings in a d-compatible topological space. Here  $F_4$  denotes the family of all real continuous functions  $f: (\mathbb{R})^4 \rightarrow \mathbb{R}$  satisfying the following properties.

$(F_k)$  there exists  $k \geq 1$  such that for every  $u \geq 0, v \geq 0$  with  $f(u, v, u, v) \geq 0$  (or)  $f(u, v, v, u) \geq 0$  we have  $u \geq kv$

$(F_u)$   $f(u, v, 0, 0) < 0$  for all  $u > 0$ .

We denote by  $\Phi$  the class of all real valued continuous functions  $\varphi: (\mathbb{R}^+)^4 \rightarrow \mathbb{R}$ , non decreasing in first argument and satisfying

$$(i) \text{ for all } u, v \geq 0, \varphi(u, v, u, v) \geq 0 \text{ (or) } \varphi(u, v, v, u) \geq 0 \Rightarrow u \geq v \quad (3.1)$$

$$(ii) \varphi(u, u, 1, 1) \geq 0 \text{ for all } u \geq 1 \quad (3.2)$$

### 4. MAIN RESULTS

Kohli, Vashistha and Kumar [4] proved the following lemma for six mappings.

**Lemma 4.1 (Lemma 4.2, Kohli, Vashistha and Kumar [4]):** Let  $(X, F, t)$  be a complete Menger space where  $T$  denotes a continuous t-norm. Further, let  $(p, hk)$  and  $(q, fg)$  be pointwise R-weakly commuting pairs of self-mappings of  $X$  satisfying

$$p(X) \subset fg(X), q(X) \subset hk(X) \quad (4.1.1)$$

$$\varphi(F_{px,qy}(\alpha t), F_{hxx,fgy}(t), F_{px,hxx}(t), F_{qy,fgy}(\alpha t)) \geq 0 \quad (4.1.2)$$

$$\varphi(F_{px,qy}(\alpha t), F_{hxx,fgy}(t), F_{px,hxx}(\alpha t), F_{qy,fgy}(t)) \geq 0 \quad (4.1.3)$$

for all  $x, y \in X, t > 0, \alpha \in (0,1)$  and for some  $\varphi \in \Phi$ .

Then the continuity of one of the mappings in the compatible pairs  $(p, hk)$  or  $(q, fg)$  on  $(X, F, T)$  implies their reciprocal continuity.

Shikha Chaudhari [14] proved the lemma by assuming  $(p, hk)$  and  $(q, fg)$  to be R-weakly commuting mappings of type (i), type (ii) and type (iii) respectively.

Now we prove an analogue of the above Lemma 4.1 for four maps.

**Lemma 4.2:** Let  $(X, F, t)$  be a complete Menger space where  $t$  denotes a continuous t-norm. Further, let  $A, S, B$  and  $T$  be self mappings on  $X$  satisfying

$$A(X) \subset T(X), B(X) \subset S(X) \quad (4.2.1)$$

$$\varphi\left(F_{Ax,By}(ht), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(ht)\right) \geq 0 \quad (4.2.2)$$

$$\varphi\left(F_{Ax,By}(ht), F_{Sx,Ty}(t), F_{Ax,Sx}(ht), F_{By,Ty}(t)\right) \geq 0 \quad (4.2.3)$$

for all  $x, y \in X, t > 0, h \in (0,1)$  and for some  $\varphi \in \Phi$ .

Moreover  $A$  and  $S$  are commuting and  $B$  and  $T$  are commuting. If

(i)  $S$  is continuous then  $(A, S)$  is reciprocally continuous.

(ii)  $T$  is continuous then  $(B, T)$  is reciprocally continuous.

**Proof:** We prove (i). The proof of (ii) is similar. Suppose  $A$  and  $S$  commute, so that  $A$  and  $S$  are compatible. Suppose  $S$  is continuous.

We shall show that  $A$  and  $S$  are reciprocally continuous.

Let  $\{x_n\}$  be a sequence in  $X$  such that  $Ax_n \rightarrow z$  and  $Sx_n \rightarrow z$  for some  $z \in X$  as  $n \rightarrow \infty$ .

Since  $S$  is continuous  $SAx_n \rightarrow Sz$ ,  $SSx_n \rightarrow Sz$ .

We show that  $ASx_n \rightarrow Az$ .

In view of compatibility of  $(A, S)$ , we have  $F_{ASx_n, SAx_n}(t) \rightarrow 1$ .

i.e.  $F_{ASx_n, Sz}(t) \rightarrow 1$

i.e.  $ASx_n \rightarrow Sz$  as  $n \rightarrow \infty$ .

In view of (4.2.1) for each  $n$ , there exists  $y_n \in X$  such that  $ASx_n = Ty_n$ .

So  $SSx_n \rightarrow Sz$ ,  $SAx_n \rightarrow Sz$ ,  $ASx_n \rightarrow Sz$  and  $Ty_n \rightarrow Sz$  as  $n \rightarrow \infty$ . (4.2.4)

Next we claim that  $By_n \rightarrow Sz$  as  $n \rightarrow \infty$ .

By putting  $x = Sx_n$  and  $y = y_n$  in (4.2.2), we get

$$\varphi \left( F_{ASx_n, By_n}(ht), F_{SSx_n, Ty_n}(t), F_{ASx_n, SSx_n}(t), F_{By_n, Ty_n}(ht) \right) \geq 0$$

Since  $\varphi$  is continuous, by (4.2.4), we have

$$\varphi \left( F_{Sz, By_n}(ht), 1, 1, F_{By_n, Sz}(ht) \right) \geq 0$$

i.e.  $F_{Sz, By_n}(ht) \geq 1$  (from (3.1))

$$\therefore F_{Sz, By_n}(ht) = 1$$

i.e.  $By_n \rightarrow Sz$  as  $n \rightarrow \infty$ .

Again putting  $x = z$  and  $y = y_n$  in (4.2.3), we get

$$\varphi \left( F_{Az, By_n}(ht), F_{Sz, Ty_n}(t), F_{Az, Sz}(ht), F_{By_n, Ty_n}(t) \right) \geq 0$$

Letting  $n \rightarrow \infty$ , we get

$$\varphi \left( F_{Az, Sz}(ht), F_{Sz, Sz}(t), F_{Az, Sz}(ht), F_{Sz, Sz}(t) \right) \geq 0$$

i.e.  $\varphi \left( F_{Az, Sz}(ht), 1, F_{Az, Sz}(ht), 1 \right) \geq 0$

By (3.1), we get  $F_{Az, Sz}(ht) \geq 1$

$$\therefore Az = Sz$$

Hence  $ASx_n \rightarrow Az$ .

This completes the proof of the lemma.

Now we introduce the notion of a DNR-function and DNR-commuting property.

**Definition 4.3:** [10] A function  $\psi: X \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is said to be a DNR function if

$\psi(x, t) > 0$  for all  $x \in X$  and  $t > 0$ .

$\Psi$  denotes the class of all DNR functions  $\psi$ .

**Example 4.4:** [10] Let  $X = \{2,3, 4, \dots\}$  with the metric  $d(x, y) = |x - y|$  and define

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq x \\ 1 & \text{if } t > y \\ \frac{t-x}{y-x} & \text{if } x < t \leq y \end{cases} \quad \text{for } x < y.$$

Clearly  $(X, F)$  is a PM-space. Define  $\psi(x, t) = \begin{cases} x & \text{if } t \leq x \\ \frac{t-1}{x} & \text{if } t > x \end{cases}$  for  $x \in [2, \infty)$

Then  $\psi$  is a DNR function.

**Definition 4.5:** [10] Suppose  $A$  and  $S$  are self maps on a PM-space  $(X, F)$ . We say that the pair  $(A, S)$  is DNR-commuting if  $z \in X$  and  $t > 0 \Rightarrow$  there exists  $\psi \in \Psi$  such that  $F_{ASz, SAz}(t) \geq F_{Az, Sz}(\psi(z, t))$ .

**Note 1:** If  $A$  and  $S$  are point wise R-weakly commuting self maps on a PM- space  $X$ , then  $A$  and  $S$  are DNR-commuting.

**Note 2:** If  $A$  and  $S$  are DNR-commuting self maps on a PM-space  $X$ , then  $A$  and  $S$  are compatible.

**Theorem 4.6:** Let  $(X, F, t)$  be a complete Menger space where  $t$  denotes a continuous t-norm. Further, let  $(A, S)$  and  $(B, T)$  be DNR-commuting pairs of self mappings on  $X$  satisfying

$$A(X) \subset T(X), B(X) \subset S(X) \tag{4.6.1}$$

$$\varphi \left( F_{Ax, By}(ht), F_{Sx, Ty}(t), F_{Ax, Sx}(t), F_{By, Ty}(ht) \right) \geq 0 \tag{4.6.2}$$

$$\varphi \left( F_{Ax, By}(ht), F_{Sx, Ty}(t), F_{Ax, Sx}(ht), F_{By, Ty}(t) \right) \geq 0 \tag{4.6.3}$$

for all  $x, y \in X, t > 0, h \in (0,1)$  and for some  $\varphi \in \Phi$ .

Moreover  $S$  commutes with  $A$  and  $T$  commutes with  $B$ . Further suppose that  $S$  and  $T$  are continuous. Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .

**Proof:** Let  $u_0 \in X$ . By (4.6.1), we define the sequences  $\{u_n\}$  and  $\{v_n\}$  in  $X$  such that for  $n = 0,1,2, \dots$

$$v_{2n+1} = Au_{2n} = Tu_{2n+1},$$

$$v_{2n+2} = Bu_{2n+1} = Su_{2n+2}$$

Now by putting  $x = u_{2n}, y = u_{2n+1}$  in (4.6.2), we get

$$\varphi \left( F_{Au_{2n}, Bu_{2n+1}}(ht), F_{Su_{2n}, Tu_{2n+1}}(t), F_{Au_{2n}, Su_{2n}}(t), F_{Bu_{2n+1}, Tu_{2n+1}}(ht) \right) \geq 0$$

$$\Rightarrow \varphi \left( F_{v_{2n+1}, v_{2n+2}}(ht), F_{v_{2n}, v_{2n+1}}(t), F_{v_{2n+1}, v_{2n}}(t), F_{v_{2n+2}, v_{2n+1}}(ht) \right) \geq 0$$

Using (3.1), we get

$$F_{v_{2n+1}, v_{2n+2}}(ht) \geq F_{v_{2n}, v_{2n+1}}(t)$$

Now by putting  $x = u_{2n+2}, y = u_{2n+1}$  in (4.6.2), we get

$$\varphi \left( F_{Au_{2n+2}, Bu_{2n+1}}(ht), F_{Su_{2n+2}, Tu_{2n+1}}(t), F_{Au_{2n+2}, Su_{2n+2}}(t), F_{Bu_{2n+1}, Tu_{2n+1}}(ht) \right) \geq 0$$

$$\Rightarrow \varphi \left( F_{v_{2n+3}, v_{2n+2}}(ht), F_{v_{2n+2}, v_{2n+1}}(t), F_{v_{2n+3}, v_{2n+2}}(t), F_{v_{2n+2}, v_{2n+1}}(ht) \right) \geq 0$$

Using (3.1), we get

$$F_{v_{2n+3}, v_{2n+2}}(ht) \geq F_{v_{2n+2}, v_{2n+1}}(t)$$

Thus for any  $n$  and  $t$ , we have  $F_{v_n, v_{n+1}}(ht) \geq F_{v_{n-1}, v_n}(t)$ .

Hence by Lemma 2.13,  $\{v_n\}$  is a Cauchy sequence in  $X$ .

Since  $X$  is complete  $\{v_n\}$  converges to  $z$ .

$\therefore$  Its subsequences  $\{Au_{2n}\}, \{Bu_{2n+1}\}, \{Su_{2n}\}$  and  $\{Tu_{2n+1}\}$  also converge to  $z$ .

Now, suppose that  $(A, S)$  is a compatible pair and  $S$  is continuous. Then by Lemma 4.2,  $A$  and  $S$  are reciprocally continuous.

Then  $ASu_{2n} \rightarrow Az$  and  $SAu_{2n} \rightarrow Sz$ .

Compatibility of  $A$  and  $S$  gives  $F_{ASu_{2n}, SAu_{2n}}(t) \rightarrow 1$ .

i.e.  $F_{Az, Sz}(t) \rightarrow 1$  as  $n \rightarrow \infty$ .

Hence  $Az = Sz$ .

Since  $A(X) \subset S(X)$ , there exists a point  $u$  in  $X$  such that  $Az = Tu$ .

Now by putting  $x = z, y = u$  in (4.6.2), we get

$$\varphi(F_{Az, Bu}(ht), F_{Sz, Tu}(t), F_{Az, Sz}(t), F_{Bu, Tu}(ht)) \geq 0$$

$$\text{i.e. } \varphi(F_{Az, Bu}(ht), 1, 1, F_{Bu, Az}(ht)) \geq 0$$

Using (3.1), we get  $F_{Az, Bu}(ht) \geq 1$  for all  $t \geq 0$

$$\Rightarrow F_{Az, Bu}(ht) = 1$$

Hence  $Az = Bu$ .

Thus  $Az = Sz = Bu = Tu$ .

Since  $A$  and  $S$  are DNR-commuting, to the pair  $(z, t)$  corresponds a  $\psi \in \Psi$  such that

$$\begin{aligned} F_{ASz, SAz}(t) &\geq F_{Az, Sz}(\psi(z, t)) \\ &= 1 \end{aligned}$$

Hence  $ASz = SAz$  and  $SAz = SSz = AAz = ASz$ .

Since  $B$  and  $T$  are DNR-commuting, we have

$$BBu = BTu = TBu = TTu.$$

Again by putting  $x = Az, y = u$  in (4.6.2), we get

$$\varphi(F_{AAz, Bu}(ht), F_{SAz, Tu}(t), F_{AAz, SAz}(t), F_{Bu, Tu}(ht)) \geq 0$$

$$\Rightarrow \varphi(F_{AAz, Az}(ht), F_{AAz, Az}(t), 1, 1) \geq 0$$

Therefore  $F_{AAz, Az}(ht) \geq 1$  for all  $t > 0$ , using (3.1)

$$\therefore F_{AAz, Az}(ht) = 1.$$

$$\Rightarrow AAz = Az \text{ and } Az = AAz = SAz.$$

$\therefore Az$  is a common fixed point of  $A$  and  $S$ .

By putting  $x = z, y = Bu$  in (4.6.3), we get

$$\varphi \left( F_{Az, BBu}(ht), F_{Sz, TBu}(t), F_{Az, Sz}(ht), F_{BBu, TBu}(t) \right) \geq 0$$

$$\Rightarrow \varphi \left( F_{Az, BAz}(ht), F_{Az, TAz}(t), 1, 1 \right) \geq 0$$

Since  $\varphi$  is non decreasing, using (3.1), we get

$$F_{Az, BAz}(t) \geq 1. (\because BAz = BTu = TBu = TAz)$$

$$\therefore Az = BAz = TAz.$$

Hence  $Az$  is a common fixed point of  $A, B, S$  and  $T$ .

Clearly,  $Az$  is the unique common fixed point of  $A, B, S$  and  $T$ .

**Note:** The theorem is valid if one of the mappings in the compatible pairs  $(A, S)$  and  $(B, T)$  is continuous instead of assuming that  $S$  and  $T$  are continuous. The proof is similar.

The following theorem is an extension to six mappings

**Theorem 4.7:** Let  $(X, F, t)$  be a complete Menger space, where  $t$  denotes a continuous t-norm. Suppose  $A, B, S, T, H$  and  $R$  are self maps on  $X$  such that  $(A, SH)$  and  $(B, TR)$  are DNR-commuting pairs of self mappings on  $X$  satisfying

$$A(X) \subset TR(X), B(X) \subset SH(X) \tag{4.7.1}$$

$$\varphi \left( F_{Ax, By}(\alpha t), F_{SHx, TRy}(t), F_{Ax, SHx}(t), F_{By, TRy}(\alpha t) \right) \geq 0 \tag{4.7.2}$$

$$\varphi \left( F_{Ax, By}(\alpha t), F_{SHx, TRy}(t), F_{Ax, SHx}(\alpha t), F_{By, TRy}(t) \right) \geq 0 \tag{4.7.3}$$

for all  $x, y \in X, t > 0, \alpha \in (0, 1)$  and for some  $\varphi \in \Phi$ .

Moreover suppose  $H$  commutes with  $A$  and  $S$  and  $R$  commutes with  $B$  and  $T$ . Suppose that  $SH$  and  $TR$  are continuous. Then  $A, B, S, T, R$  and  $H$  have a unique common fixed point in  $X$ .

**Proof:** Write  $SH = P$  and  $TR = Q$ .

By hypothesis,  $P$  and  $Q$  are continuous.

Thus by Theorem 4.6,  $A, B, P$  and  $Q$  have a unique common fixed point  $z$  in  $X$ .

$$\text{i.e. } Az = Bz = Pz = Qz = z.$$

Take  $x = Hz, y = z$  in (4.7.2). We get

$$\varphi \left( F_{AHz, Bz}(\alpha t), F_{SHHz, TRz}(t), F_{AHz, SHHz}(t), F_{Bz, TRz}(\alpha t) \right) \geq 0$$

$$\varphi \left( F_{HAz, z}(\alpha t), F_{HSHz, Qz}(t), F_{HAz, HSHz}(t), F_{z, Qz}(\alpha t) \right) \geq 0$$

$$\varphi \left( F_{Hz, z}(\alpha t), F_{Hz, z}(t), F_{Hz, Hz}(t), F_{z, z}(\alpha t) \right) \geq 0$$

$$\text{Hence } \varphi \left( F_{Hz, z}(\alpha t), F_{Hz, z}(t), 1, 1 \right) \geq 0$$

$$\Rightarrow F_{Hz, z}(t) \geq 1 \text{ for every } t > 0 \text{ ( } \because \varphi \text{ is non decreasing in its first co ordinate)}$$

$$\Rightarrow Hz = z$$

$\Rightarrow z$  is a fixed point of  $H$ .

Now  $z = Pz = SHz = Sz$ .

$\therefore z$  is also a fixed point of  $S$ .

Now take  $x = z$  and  $y = Tz$  in (4.7.2). We get

$$\varphi \left( F_{Az, BTz}(\alpha t), F_{SHz, TRTz}(t), F_{Az, SHz}(t), F_{BTz, TRTz}(\alpha t) \right) \geq 0$$

$$\Rightarrow \varphi \left( F_{z, TBz}(\alpha t), F_{z, TRz}(t), F_{z, z}(t), F_{TBz, TRz}(\alpha t) \right) \geq 0$$

$$\Rightarrow \varphi \left( F_{z, Tz}(\alpha t), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(\alpha t) \right) \geq 0$$

$$\Rightarrow \varphi \left( F_{z, Tz}(\alpha t), F_{z, Tz}(t), 1, 1 \right) \geq 0$$

$$\Rightarrow F_{z, Tz}(t) \geq 1 \text{ for every } t > 0 \text{ (} \because \varphi \text{ is non decreasing in its first co ordinate)}$$

$$\Rightarrow Tz = z.$$

$\therefore z$  is also a fixed point of  $T$

Now  $z = Qz = TRz = RTz = Rz$ .

$\therefore z$  is also a fixed point of  $R$

Hence  $z$  is a common fixed point of  $A, B, S, T, R$  and  $H$ .

Suppose  $x$  is also a fixed point of  $A, B, S, T, R$  and  $H$ .

Then  $Pz = SHz = S(Hz) = Sz = z$

and  $Qz = TRz = T(Rz) = Tz = z$

Similarly  $Qx = Px = x$ . Thus  $x$  and  $z$  are common fixed points of  $A, B, P$  and  $Q$ .

Hence by Theorem 4.6,  $z = x$ .

Thus  $A, B, S, T, R$  and  $H$  have unique fixed point.

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