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On the Reverse of determining the n^{th} Numeric Palindrome

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ABSTRACT

This paper mainly gives a method/algorithm on how to determine the reverse of the method of determining the n^{th} numeric palindrome .That is given a numeric palindrome we determine n. Also it gives a method/algorithm on how to find the number of palindromic number less than or equal to $\boldsymbol{x}$, *where* $x \in \boldsymbol{N}$.

INTRODUCTION

1,2,3,4,5,6,7,8,9,11,22,33,44,55,66,77,88,99:These are the first 18 numeric palindromes (Wolfram Math World [1] *a number (in some base b) that is the same when written forwards or backwards).* We denote *P* to be the set of all numeric palindromes in the set of *N*. Then, *P* ={1,2,3,4,5,6,7,8,9,11,22,33,44,55,66,77,88,99,101,…}. In *P* the 1st numeric palindrome is 1, the 2nd is 2 and 101 is the 19th. With this we define the "rank" of the numeric palindrome.

Definition 1: *Given* ***P****, the* ***rank*** *of a (denoted by* ***R****(a)), where a* ∈ ***P*** *is n, provided that a is the* n^{th} *numeric palindrome in* ***P.***

Thus, the rank of 1 is 1, 2 is 2 and 101 is 19.

Note that, from the list of palindromic numbers above, there are **9** one- digit numeric palindromes and **9** two- digit numeric palindromes. We call **9** to be the *cardinality* of the one- digit numeric palindrome and similarly to two- digit numeric palindrome.

Definition 2*: The* ***cardinality*** *of an r-digit palindromic number denoted by C (r) is the number or count of r digit palindromic number in* ***P****.*

From definition 2, *C* (1) = 9 and *C* (2) = 9 also. From Fuehrer [2] the general formula for finding *C* (r) is given by the formula:

$$C\ (r) = (10^{\frac{r}{2}-1})(9), if\ r\ is\ even$$

$$C\ (r) = \left(10^{\frac{r+1}{2}-1}\right)(9), if\ r\ is\ odd$$

But, for easy reference the two formulas above can be made simple by the formula:

$$\boldsymbol{C\ (r) = 9\left(10^{\left[\!\left[\frac{r-1}{2}\right]\!\right]}\right)} \qquad (1)$$

In this paper, we will usually use definition 1 and 2 and formula **[1].**

MAIN RESULTS

Rank of numeric palindrome with respect to their number of digits (r).

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Consider the table of numeric palindromes below:

(r)	*C*(r)	Numeric palindromes
1	**9**	**1,2,3,4,5,6,7,8,9**
2	**9**	**11,22,33,44,55,66,77,88,99**
3	**90**	**101,111,121,131,141,151,161,171,181,191 202,212,222,232,242,252,262,272,282,292 ⋮ 909,919,929,939,949,959,969,979,989,999**
4	**90**	**1001,1111,1221,1331,1441,1551,1661,1771,1881,1991 2002,2112,2222,2332,2442,2552,2662,2772,2882,2992 ⋮ 9009,9119,9229,9339,9449,9559,9669,9779,9889,9999**
5	**900**	**10001,10101,10201,10301,10401,10501,10601,10701,10801,10901 11011,11111,11211,11311,11411,11511,11611,11711,11811,11911 12021,12121,12221,12321,12421,12521,12621,12721,12821,12921 ⋮ 99099,99199,99299,99399,99499,99599,99699,99799,99899,99999**

We start for *r* = 3, for the rank of numeric palindrome for *r* = 1 and 2 are extremely trivial!

For *r* = 3, let us take 101 to be the numeric palindrome under consideration. The rank of 101 with respect to 3 digit numeric palindrome is 1. If our numeric palindrome of 3 digit is 232 then its rank is 14 (see table). Lastly if it is 999 then its rank is 90 for sure because it is the last 3 – digit numeric palindrome.

Notice that the unit digit of the rank of a 3 digit numeric palindrome is equal to the unit digit of middle digit (**m**) + 1.

For 232 we already know that the unit digit of its rank is 4 and its rank is 14. So we need to add 10 to 4 so that we can get 14 which is the rank of 232 with respect to 3 digit numeric palindrome. Where did we get 10? We get 10 from the first digit of 232 which is 2 by applying the formula $10(2-1)$.Thus to find the rank of the given numeric palindrome with *r* = 3 one can use the formula:

$$R(a) = 10\,(f_1\ -1) + m + 1 \qquad (2)$$

Where, ***a*** is the given numeric palindrome, $\boldsymbol{f_1}$ is the first digit of *a* and ***m*** is the middle digit of *a*.

Let us apply [2] in finding the rank of 999 and 101.

$$R(999) = 10(9-1) + 9 + 1 = 10(8) + 10 = 90$$

$$R(101) = 10(1-1) + 0 + 1 = 10(0) + 1 = 1$$

For *r* = 5, let us consider *a* = 12 121. We know from the table above that *R*(*a*) with respect to 5 – digit numeric palindrome is 22. Note again that the unit digit of the rank of *a* is the unit digit of the sum of *m*+1 which is 2. To get 20 we use the formula 10 $(f_2 - 10) = 10(12-10) = 20$.

Let *a* = 99 999, we will find *R* (*a*) by using the formula being describe by the statement above.

$$R(99\,999) = 10(99-10) + 9 + 1 = 890 + 10 = 900$$

Which is exactly the rank of *a* in *r* = 5.

Formula [2] can be extended for any odd r greater than or equal to 3. We will now state the general formula in finding the rank of a numeric palindrome with respect to its number of digits for odd *r*.

For any odd r – digit numeric palindrome a starting from 3, the rank of a, R (a) is given by:

$$\boldsymbol{R\,(a) = 10\left(f_{\frac{r-1}{2}} - 10^{\frac{r-3}{2}}\right) + (m+1)} \qquad (3)$$

Where *a* is the given numeric palindrome; $\boldsymbol{f_{\frac{r-1}{2}}\ is\ the\ first\ \frac{r-1}{2}\ digit\ of\ a}$ **; and *m* is the middle digit of *a*.**

Let us now consider the case when *r* is even.

For r = 4, let us take 1 001 to be the numeric palindrome under consideration. The rank of 1 001 with respect to 4 digit numeric palindrome is 1. If our numeric palindrome of 4 digits is 9 229 then its rank is 83. (see table) Lastly if it is 9 999 then its rank is 90, for sure because it is the last 4 – digit numeric palindrome.

Note that for r being even, we have a 2 – digit middle number with the same numerical value. Also same as for odd case, the unit digit of the rank of our r – digit numeric palindrome is equal to the unit digit of the sum of m + 1, where m is one of the 2 – digit middle number.

For a = 9 229, we already knew its rank with respect to 4 – digit numeric palindrome which is 83. And from our note earlier that the unit digit of the rank of our r – digit numeric palindrome is equal to the unit digit of the sum of m + 1, where m is one of the 2 – digit middle number: which is 3, the only problem now is how to find the remaining number 80. But same as for odd case, 80 can be get by applying the formula $10(f_1 - 1)$, applying the formula we have: $10(9 - 1) = 80$. Thus, for r = 4 the rank of a, R (a) can be found by using:

$$R(a) = 10(f_1 - 1) + m + 1 \tag{4}$$

Where, $\boldsymbol{a}$ is the given numeric palindrome; $\boldsymbol{f_1}$ is the first digit of a; and $\boldsymbol{m}$ is one of the middle digit of a. This formula can be extended for any even number r greater than 4.

We will now state the general formula in finding the rank of a numeric palindrome with respect to its number of digits for even r.

For any even r – digit numeric palindrome a starting from 4, the rank of a, R (a) is given by:

$$\boldsymbol{R\ (a) = 10\left(f_{\frac{r-2}{2}} - \ 10^{\frac{r-4}{2}}\right) + (m+1)} \tag{5}$$

Where a is the given numeric palindrome; $\boldsymbol{f_{\frac{r-2}{2}}\ is\ the\ first\ \frac{r-2}{2}\ digit\ of\ a}$; and m is one of the middle digit of a.

Let us show that the formula works by applying it with a = 9 999 and a = 998 899

$$R(9\,999) = 10(9 - 1) + 9 + 1 = 10(8) + 10 = 90$$

$$R(998\,899) = 10(99 - 10) + 8 + 1 = 10(89) + 9 = 899$$

Note that our answers are correct for 9 999 is the last 4 – digit numeric palindrome and 998 899 is the 2nd to the last 6 – digit numeric palindrome.

Rank of numeric palindrome with respect to *P* R_P *(a)*.

To determine the rank of a given numeric palindrome with respect to the set of numeric palindrome ***P,*** we will simply use formula **[1]**, **[3]** and **[5]**.

STEPS:

1. Determine the digit (r) of the numeric palindrome and use [3] or [5] to find its rank with respect to (r).
2. Using formula [1] determine the number of palindromes for k = 1, 2, …, r-2, r-1.
3. Get the sum of the numbers obtain from 1 and 2 then we are done.

The steps above can be put into a formula:

$$R_P(a) = \left[\sum_{k=1}^{r-1}(10^{[\![\frac{k-1}{2}]\!]})(9)\right] + 10\left(f_{\frac{r-1}{2}} - \ 10^{\frac{r-3}{2}}\right) + (m+1)\text{ , if r is odd} \tag{6}$$

$$R_P(a) = \left[\sum_{k=1}^{r-1}(10^{[\![\frac{k-1}{2}]\!]})(9)\right] + 10\left(f_{\frac{r-2}{2}} - \ 10^{\frac{r-4}{2}}\right) + (m+1)\text{, if r is even} \tag{7}$$

Example: Among the list of numeric palindromes, what is the rank of 11 111 111 111?

Solution: We are given, 11 – digit numeric palindrome. We will use equation [3] to find its rank with respect to its digit. Since 11 is odd, using [3] we have:

$$R(11\,111\,111\,111) = 10(11\,111 - 10^4) + 1 + 1$$

$$R(11\,111\,111\,111) = 10(11\,111 - 10\,000) + 2$$

$$R(11\,111\,111\,111) = 11\,112$$

Thus, 11 111 111 111 is the 11 112th 11 digit numeric palindrome. Using formula [1] we have a sum total of:

$$9 + 9 + 90 + 90 + 900 + 900 + 9\,000 + 9\,000 + 90\,000 + 90\,000 = 199\,998$$

numeric palindromes from $r = 1$ up to $r = 11\text{-}1 = 10$. Getting the sum we then have 211 110. Thus, 11 111 111 111 is the 211 110th numeric palindrome $\boldsymbol{R_P}$ ***(11 111 111 111) =211 110***.

On determining the number of numeric palindrome less than or equal to $x \in N$

The problem of determining the number of numeric palindrome less than or equal to $x \in N$ can be simplified by using our early result. This suggests that given x we will just simply find for the greatest palindrome less than x say a then find the rank of a with respect to the set of numeric palindromes.

Let us start with $x = 146$, the greatest numeric palindrome less than 146 is 141 and R_P (141) = 23, thus there are 23 numeric palindromes less than or equal to 146. If $x = 2249$, the greatest numeric palindrome less than 2 249 is 2 222 and using the formula earlier we get R_P (2 249) =121, thus there are 121 numeric palindrome less than or equal to 2 249.

It is not so obvious but true that the number of numeric palindrome less than or equal to x only depends to the numbers comprising a and the number of digits of a. This is given by the algorithm:

For odd $r \geq 3$. The number of numeric palindrome less than or equal to x is given by:

$$\left[\sum_{k=1}^{r-1}\left(10^{\left[\!\left[\frac{k-1}{2}\right]\!\right]}\right)9\right] + 10\left(f_{\frac{r-1}{2}} - 10^{\frac{r-3}{2}}\right) + m + i \tag{8}$$

Where *i* takes the value 0 or 1 depending on the following criteria:

If m + 1digit > m – 1 digit take i = 0 (9)

If m + 1 digit < m – 1 digit take i = 1 (10)

If m + 1 digit = m-1 digit , consider m + 2 and m – 2 digit and back to [9] and [10] with m + 2 and m – 2 replacing m + 1 and m – 1. And so on and so forth.

For even $r \geq 4$. The number of numeric palindrome less than or equal to x is given by:

$$\left[\sum_{k=1}^{r-1}\left(10^{\left[\!\left[\frac{k-1}{2}\right]\!\right]}\right)9\right] + 10\left(f_{\frac{r-2}{2}} - 10^{\frac{r-4}{2}}\right) + m_l + i \tag{11}$$

Where *i* takes the value 0 or 1 depending on the following criteria:

If $m_l > m_r$ take i = 0 (12)

If $m_l < m_r$ take i = 1 (13)

If $m_l = m_r$, consider m_{l+1} and m_{r-1} with m_{l+1} and m_{r-1} replacing m_l and m_r and back to [12] and [13]. And so on and so forth. Where m_l middle left digit and m_r is the middle right digit.

Let us apply the method above to find the number of numeric palindrome less than or equal to $x = 2\,249$.

Using formula [11] and note that 2 249 is a 4 – digit number we have:

$$9 + 9 + 90 + 10\,(2 - 10^0) + 2 + 1 \text{ (since } 2 < 4) = 121 \text{ which is the answer above.}$$

Find the number of numeric palindrome less than or equal to $x = 123\,456\,789\,101\,112$.

Note that we have a 15 – digit number with $m = 8$. Using formula [8] we have:

$$9 + 9 + 90 + 90 + 900 + 900 + 9{,}000 + 9{,}000 + 90{,}000 + 90{,}000 + 900{,}000 + 900{,}000 + 9{,}000{,}000 + 9{,}000{,}000 + 8 + 1(because\ 7 < 9) = 18{,}200{,}007.$$

Thus, there are 18,200,007 numeric palindrome less than or equal to 123 456 789 101 112.

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