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COMMON FIXED POINTS IN NORMED SPACES USING α- PROPERTY VIA WEAKLY-BIASED AND (OWC) MAPS

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ABSTRACT

*In this paper, we will prove and show by an Example that, the condition of weakly compatibility (and (owc), as well) implies to weakly S-biased maps and weakly A-biased maps, but not conversely. By using this fact, we will prove two types of fixed point results: **part I**, for weakly biased maps; and **part II**, for (owc) maps. The results of part I (Theorem 2.1 and Theorem 2.4) are the generalization of the Theorems of Ciric and Um´e [2], while the result of part II (Theorem 2.7) is the generalization of Theorem 1 of Pathak and Verma [13].*

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1. INTRODUCTION

By generalizing the concept of commuting mappings, Sessa [15] introduced the concept of weakly commuting mappings. Various commuting type mappings are compatible maps [3], compatible maps of type (A) [5], compatible maps of type (B) [12], compatible maps of type (C) [10], compatible maps of type (P) [9], compatible maps of type (T) [11] and R-weakly commuting [8] etc. These are generalizations of weakly commuting mapping of Sessa [15]. The concept of weakly compatibility was introduced by Jungck [4], as a generalization of compatibility [3]. The author of this paper has shown (see, [13]) that all the above compatibility types imply to weakly compatibility. Recently, Al-Thagafi [1] introduced and Jungck-Rhoades [7] used the concept of occasionally weakly compatible (owc) maps, as a generalization of weakly compatible maps. Besides, the concept of “biased” maps of Jungck and Pathak [6] was further generalized to weakly-biased maps. It is shown in Proposition 1.1 [6] that every biased map is weakly biased but not conversely. We will discuss about weakly compatibility, (owc) maps and weakly biased maps.

1.1. The (owc) maps and weakly biased maps. Before we show that the (owc)-maps imply to weakly S-biased (and, weakly A-biased) maps, we need to define it.

Definition 1. [4] Let A and S be two self-maps of a metric space (X, d). The mappings A and S are said to be weakly compatible if they commute at their coincidence points, i.e,

$$ASx = SAx, \textit{ for all } Ax = Sx, \text{ where } x \in C(A, S), \tag{1.1}$$

where C(A, S) denotes the set of coincidence points of X.

Definition 2. [7] The mappings A and S of a metric space (X, d) are said to be occasionally weakly compatible (owc) mappings if and only if

$$Ax = Sx \text{ and } ASx = SAx, \text{ for some } x \in C(A, S). \tag{1.2}$$

Every weakly compatible mapping is (owc) but not conversely ([7]).

Definition 3. [6] Let A and S be two self-maps of a metric space (X, d). The pair (A, S) is called S-biased if, whenever there exists a sequence $\{x_n\}$ in X such that $Ax_n \to t$, $Sx_n \to t$ as $n \to \infty$, then

$$L d(SAx_n, Sx_n) \le L d(ASx_n, Ax_n), \text{ where } L = \lim\inf \text{ or } L = \lim\sup. \tag{1.3}$$

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Definition 4. [6] Let A and S be two self-maps of a metric space (X, d). The pair (A, S) is called weakly S-biased, if and only if

$$Ap = Sp \textit{ implies } d(SAp, Sp) \leq d(ASp, Ap). \quad (1.4)$$

Every S-biased map is weakly S-biased (see, Proposition 1.1 [6]). By interchanging the role of mappings A and S, we can define A-biased and weakly A-biased.

Now, we prove by Lemma 1.1 and Lemma 1.2, and show by Example1.3 that, every (owc) maps is weakly S-biased as well as weakly A-biased maps. We underline by some assertion below, the importance of notions of weakly compatible maps and (owc) maps, even if it is weakly S-biased map (or, weakly A-biased map).

Lemma 1.1. Let A and S be two self-maps of a metric space (X, d). If the pair (A, S) is (owc) then it is weakly S-biased maps as well as weakly A-biased maps.

Proof. Let A and S be a pair of (owc) maps then, by definition, Ap = Sp and ASp = SAp for some p ε C (A, S). So that, whenever p ε C(A, S), we have ASp = SAp, and so that

$$d(SAp, Sp) = d(ASp, Ap). \quad (1.5)$$

This relation of equality in eq. (1.5) is always true for weakly S-biased as well as weakly A-biased maps. Thus the Lemma follows.

Lemma 1.2. Let A and S be two self-maps of a metric space (X, d). If the pair (A, S) is weakly S-biased (or, weakly A-biased) then it need not imply (owc) maps.

Proof. Suppose that A and S are weakly S-biased maps then, whenever Ap = Sp = t (say), for all p ε X, we have

$$d(SAp, Sp) \leq d(ASp, Ap) \textit{ implies } d(St, t) \leq d(At, t) \quad \textit{not implies} \;\; At = St. \quad (1.6)$$

Here observe in (1.6) that, if St = t *not equal to* At, then (A, S) is neither weakly compatible nor (owc). Hence weakly S-biased maps need not imply weakly-compatible, or (owc). Symbolically, weakly S-biased not imply to weakly compatible. This completes the proof of this Lemma.

This assertion also indicates that:

(1) If S has a fixed point and weakly S-biased with S, then it neither guarantees the weakly compatibility, nor (owc), nor the existence of common fixed point; which, in other word, shows the importance of mappings to be weakly compatibility, and (owc) to have a common fixed point.

(2) If A has a fixed point and weakly S-biased with A, then it compels S, to have a common fixed point. That is, weakly S-biased with fixed point of A implies the existence of common fixed point with S.

(3) If C (A, S) is a singleton set, then we have Ap = Sp = t, for some p ε C (A, S).

Now, if the pair (A, S) is weakly compatible or (owc), then from commutativity, St =SAp=ASp=At; whence tεC(A, S). Thus p = t is the unique common fixed point of A and S. On the other hand, if C(A, S) is a singleton set and the pair (A, S) is weakly S-biased then from (1.6), we can't confirm the uniqueness of fixed-point. Similar argument can be stated for weakly A-biased maps.

The following example illustrates the above Lemma:

Example 1.3. Let A, S:[0, 1]→[0, 1] be two self-maps of a metric space with the usual metric d. Define A and S by: Ax=1, if xε Q∩[0, 1], Ax=0, if xε (R − Q)∩(0, 1), and Sx = 0, if 0 ≤ x < 1, and Sx=1, if x=1.

Observe that, A and S have points of coincidence x_1 ε (R−Q) ∩ (0, 1) and $x_2 = 1$. Note that, $ASx_1=1$ and $SAx_2=0$, i.e., (A, S) is not weakly compatible and also not (owc); but $d(SAx_1, Sx_1) = 0 < d(ASx_1, Ax_1) = 1$ shows that it is weakly S-biased.

Sedghi and Shobe [14] defined a new binary operation (◊) and property-α as follows:

1.2. Binary operation (◊) and property-α

Throughout this paper, let N denotes the set of all natural numbers, R the set of all real numbers and R^+ the set of all positive real numbers. Shedghi and Shobe defined the following binary operation:

Definition 5. [14] Let ◊: $R^+ \times R^+ \to R^+$ be a binary operation satisfying the following conditions:
(i) ◊ is associative and commutative,
(ii) ◊ is continuous.

Some examples of binary operation ◊ are a◊b=*max* {a, b}, a◊b=ab/ [*max*{a, b, 1}], a◊b= a+b, a◊b=a.b and a◊b=ab+a+b, for all a, b ε R^+.

Definition 6. [14] The binary operation ◊ is said to satisfy property-α if there exists a positive real number α such that

$$a\Diamond b \le max\{a, b\}, \ \ for\ all\ a, b\ \varepsilon\ R^+. \tag{1.7}$$

In the first part of this paper, we will generalize the results of Theorem 2.1 and Theorem 2.5 of Ciri´c and Um´e [2]. In the second part, we will generalize the main result of Theorem 1 of Pathak and Verma [13]

2. MAIN RESULTS

Part- I: Fixed point theorems for weakly-biased maps

Theorem 2.1. *Let A, B, S and T be four self-mappings of a normed space X, and let C be a closed and convex subset of X, satisfying the following condition:*

$$||Sx-Ty||^p \le a||Ax-By||^p \Diamond b\ max\{\lambda||Sx-By||^p, \lambda||Ty-Ax||^p\} \Diamond c\ min\{||Sx-Ax||^p, ||Ty-By||^p\} \tag{2.1}$$

for all x, yε C, where 0<a<1, 0<b<1, 0<λ<1, p>0, c≥0, 0<aα<1, 0<bαλ<1 and ◊ satisfies property-α and suppose that

$$A(C)\ superset\ (1-k)A(C)+kS(C), \quad B(C)\ superset\ (1-k')B(C)+k'T(C) \tag{2.2}$$

for some fixed k, k' such that 0<k<1, 0<k'<1. If for some x_0 ε C, a sequence {xn} in C defined inductively for n = 0, 1, 2, 3, ... by

$$Ax_{2n+1}= (1-k)Ax_{2n}+kSx_{2n},\ Bx_{2n+2}=(1-k')Bx_{2n+1} + k'Tx_{2n+1} \tag{2.3}$$

converges to a point z ε C. If A and B are continuous at z, and if (A, S) is weakly A-biased and (B, T) is weakly B-biased, then A, B, S and T have a unique common fixed point at w=Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Proof. Let us show that Az = Bz = Sz = Tz. Since A is continuous, letting n→∞ in the relation $kSx_{2n} = Ax_{2n+1}-(1-k)Ax_{2n}$, we get $lim_{n\to\infty}Sx_{2n}$=Az. Similarly, the continuity of B yields $lim_{n\to\infty}Tx_{2n+1}$=Bz. Assume that Az≠Bz. Then, putting x_{2n} for x and x_{2n+1} for y in (2.1), and since ◊ is continuous, so letting n→∞, we get

$$\begin{aligned}||Az-Bz||^p &\le (a||Az-Bz||^p)\Diamond(b\lambda||Az-Bz||^p)\Diamond(c.0)\\ &\le \alpha\ max\{a||Az-Bz||^p, b\lambda||Az-Bz||^p, 0\}\\ &< ||Az-Bz||^p\end{aligned}$$

a contradiction. Thus Az=Bz. If Az ≠ Tz, then putting x_{2n} for x and z for y in (2.1), and letting n→∞ we get

$$\begin{aligned}||Az-Bz||^p &\le (a.0) \Diamond (b\lambda||Az-Bz||^p) \Diamond (c.0)\\ &\le \alpha\ max\{0, b\lambda||Az-Bz||^p, 0\}\\ &< ab\lambda||Az-Bz||^p\end{aligned}$$

a contradiction. Thus Az = Tz. Similarly, Sz = Bz. Hence,

$$Az = Bz = Sz = Tz = w. \tag{2.4}$$

Next, since (A, S) is weakly A-biased; we have by definition, ||ASz-Az|| ≤ ||SAz−Sz||, that is ||Aw−w || ≤ ||Sw−w||. We show that Sw = w, and hence Aw=w. For, putting w for x and z for y in (2.1), we obtain

$$\begin{aligned}||Sw-w||^p = ||Sw-Tz||^p &\le (a||Aw-w||^p)\Diamond(b\ max\{||Sw-w||^p, ||Aw-w||^p\})\Diamond(c.0)\\ &\le \alpha\ max\{a||Aw-w||^p, b||Sw-w||^p, 0\}\\ &< ||Sw-w||^p\end{aligned}$$

a contradiction. Thus Sw = w = Aw. Similarly, Tw = w = Bw. Hence, we have

$$Aw = Bw = Sw = Tw = w \quad (2.5)$$

Further, if A is continuous at w, then we show that S is also continuous at w. For, let $\{y_n\}$ be an arbitrary sequence in C converging to w. Put y_n for x and w for y in (2.1), we get

$$\|Sy_n-Tw\|^p \le (a\|Ay_n-Bw\|^p)\Diamond(b\lambda\, max\{\|Sy_n-Bw\|^p, \|Ay_n-Tw\|^p\})\Diamond(c.0)$$

i.e,

$$\|Sy_n-Sw\|^p \le \alpha\, max\{a\|Ay_n-Aw\|^p, b\lambda\, max\{\|Sy_n-Sw\|^p, \|Ay_n-Aw\|^p\}, 0\}$$

If $\|Ay_n-Aw\|^p$ is the '*max*' then, since A is continuous, letting $n\to\infty$, we get

$$lim_{n\to\infty}\|Sy_n-Sw\|^p \le a.0, \text{ that is } Sy_n\to Sw.$$

If $\|Sy_n - Sw\|^p$ is '*max*' then letting $n\to\infty$, we get

$$lim_{n\to\infty}\|Sy_n - Sw\|^p \le \alpha\, max\{0, b\lambda\, lim_{n\to\infty}\|Sy_n-Sw\|^p, 0\} = \alpha b\lambda\, lim_{n\to\infty}\|Sy_n-Sw\|^p,$$

a contradiction. Thus $Sy_n\to Sw$. Hence S is continuous. Similarly, if B is continuous at w then T is continuous at w. The uniqueness of common fixed point follows easily, by using (2.1). This completes the proof.

Corollary 2.2. *Let A, B, S and T be four self-mappings of a normed space X. Let C be a closed and convex subset of X satisfying the following condition (1^M):*

$$\|Sx-Ty\|^p \le max[a\|Ax-By\|^p, b\, max\{\lambda\|Sx-By\|^p, \lambda\|Ty-Ax\|^p\}, c\, min\{\|Sx-Ax\|^p, \|Ty-By\|^p\}]$$

for all x, y ε C, where $0<a<1$, $0<b<1$, $0<\lambda<1$, $p>0$ and $c\ge0$ such that $max\{a, b\lambda, c\}<1$; and the set-inclusion eq. (2.2) satisfy with $0<k<1$, $0<k'<1$. Further, for some x_0 ε C, the sequence $\{x_n\}$ in C defined inductively by (2.3), converges to a point z ε C. If A and B are continuous at z, and if (A, S) is weakly A-biased and (B, T) is weakly B-biased, then A, B, S and T have a unique common fixed point at w=Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Proof. If $u\Diamond v=min\{u, v\}$, for each u, v ε R^+, then for any α with $\alpha\ge1$, we have

$u\Diamond v=min\{u, v\}\le\alpha\, max\{u, v\}$. Hence ◊ satisfy property-α. Similarly, for three co-ordinates

$u\Diamond v\Diamond w = min\{u, v, w\} \le \alpha\, max\{u, v, w\}$, where $\alpha \ge 1$.

Putting α=1, we get $u\Diamond v\Diamond w \le max\{u, v, w\}$. Thus, if $0<max\{a, b\lambda, c\}<1$, then all the conditions of **Theorem 2.1** hold. Therefore, A, B, S and T have a unique common fixed point at w = Tz. This completes the proof.

Corollary 2.3. *Let A, B, S and T be four self-mappings of a normed space X. Let C be a closed and convex subset of X satisfying the following condition (1^+):*

$$\|Sx-Ty\|^p\le a\|Ax-By\|^p+b\, max\{\lambda\|Sx-By\|^p, \lambda\|Ty-Ax\|^p\}+c\, min\{\|Sx-Ax\|^p, \|Ty-By\|^p\}$$

for all x, y ε C, where $0<a<1$, $0<b<1$, $0<\lambda<1$, $c\ge0$ and $p>0$ such that $0<a+b\lambda+c<1/2$; and the set -inclusion relations satisfy with $0<k<1$, $0<k'<1$. If for some x_0 ε C, the sequence $\{x_n\}$ in C defined inductively by (2.3) converges to a point z ε C. If A and B are continuous at z, and if (A, S) is weakly A-biased and (B, T) is weakly B-biased, then A, B, S and T have a unique common fixed point at w = Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Proof. Define $a\Diamond b = a+b$ for each a, bε R_+. Then for $\alpha\ge 2$, we have $a\Diamond b\le\alpha\, max\{a, b\}$. Thus, ◊ satisfy property -α. If α=2, we get $a\Diamond b\le\alpha\, max\{a, b\}$. Thus if $0<2(a+b\lambda+c)<1$, then all the conditions of **Theorem 2.1** hold. Therefore, A, B, S and T have a unique common fixed point at w=Tz.

Now, we prove our second result for the inequality which uses an upper semi-continuous function φ defined over the set of non-negative real numbers such that $\varphi(t)<t$ for each $t > 0$.

Theorem 2.4. *Let A, B, S and T be four self-mappings of a normed space X and let C be a closed and convex subset of X satisfying the following condition:*

$$||Sx-Ty||^p \le \varphi([\{2a||Ax-By||^{2p}\}/\{||Sx-By||^p+||Ty-Ax||^p\}]\lozenge[b\,max\{||Sx-By||^p, ||Ty-Ax||^p\}]\lozenge[c\,min\{||Sx-Ax||^p, ||Ty-By||^p\}]) \quad (2.6)$$

for all x, y ε C for which $||Sx-By||^p+||Ty-Ax||^p \neq 0$, where $0<a<½$, $0<b<1$, $p>0$, $c\ge 0$, $0<2a\alpha<1$, $0<b\alpha<1$ and ◊ satisfies property-α; and $\varphi:R\to R_+$ is an u.s.c. function such that $\varphi(t)<t$ for each $t>0$. Suppose that the set inclusion relation (2.2) and the inductive sequence relation (2.3) satisfy. If A and B are continuous at z, and if (A, S) is weakly A-biased and (B, T) is weakly B-biased, then A, B, S and T have a unique common fixed point at w=Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Proof. As in **Theorem 2.1** we can prove that

$$lim_{n\to\infty}Ax_n=lim_{n\to\infty}Sx_{2n}=Az, \qquad lim_{n\to\infty}Bx_n=lim_{n\to\infty}Tx_{2n+1}=Bz \quad (2.7)$$

If we assume that $Az \neq Bz$, then for large enough n, $||Sx_{2n}-Bx_{2n+1}||>0$. Thus, from (2.6) we have

$$||Sx_{2n}-Tx_{2n+1}||^p \le \varphi([\{2a||Ax_{2n}-Bx_{2n+1}||^{2p}\}/\{||Sx_{2n}-Bx_{2n+1}||^p+||Tx_{2n+1}-Ax_{2n}||^p\}]$$
$$\lozenge[b\,max\{||Sx_{2n}-Bx_{2n+1}||^p, ||Tx_{2n+1}-Ax_{2n}||^p\}]$$
$$\lozenge[c\,min\{||Sx_{2n}-Ax_{2n}||^p, ||T\,x_{2n+1}-Bx_{2n+1}||^p\}])$$

Since ◊ is continuous, and $\varphi(t)<t$, by making $n\to\infty$ it yields

$$||Az-Bz||^p \le \varphi(2a||Az-Bz||^p \lozenge b||Az-Bz||^p \lozenge [c.0])$$
$$< 2a||Az-Bz||^p\lozenge b||Az-Bz||^p\lozenge 0$$
$$\le \alpha\,max\{2a||Az-Bz||^p, b||Az-Bz||^p, 0\}$$
$$= ||Az-Bz||^p\,max\{2a\alpha, b\alpha, 0\} \quad \text{(by taking } ||Az-Bz||^p \text{ common)}$$
$$< ||Az-Bz||^p, \quad \text{(as } 0<2a\alpha<1, 0<b\alpha<1)$$

a contradiction. So that Az = Bz. Now, if we assume that $||Az-Tz|| > 0$, then for enough large n, $||Ax_{2n}-Tz||>0$. Thus, putting x_{2n} for x and z for y, in (2.6) we get

$$||Sx_{2n}-Tz||^p\le\varphi([\{2a||Ax_{2n}-Bz||^{2p}\}/\{||Sx_{2n}-Bz||^p+||Tz-Ax_{2n}||^p\}]$$
$$\lozenge[b\,max\{||Sx_{2n}-Bz||^p, ||Tz-Ax_{2n}||^p\}]\lozenge[c\,min\{||Sx_{2n}-Ax_{2n}||^p, ||Tz-Bz||^p\}])$$

since ◊ is continuous, on letting $n\to\infty$, it yields

$$||Az-Tz||^p \le \varphi(0\lozenge[b||Tz-Az||^p]\lozenge 0) < \alpha||Tz-Az||^p\,max\{0, b, 0\} < ||Tz-Az||^p,$$

a contradiction. Thus Az = Tz. Similarly Sz = Bz. Therefore, we have

$$Az = Bz = Sz = Tz = w. \quad (2.8)$$

Since the pair (A, S) is weakly A-biased and the pair (B, T) is weakly B-biased, as in **Theorem 2.1**, we can show that

$$Aw = Bw = Sw = Tw = w. \quad (2.9)$$

Now we prove that, if A and B are continuous at w, then S and T are continuous at a common fixed point w. We show that

$$||Sx-Sw||\le ||Ax-Aw|| \quad (2.10)$$

for all x ε C. Suppose that $||Sx-Sw|| > ||Ax-Aw||$. Then from (2.6), we get

$$||Sx-Sw||^p = ||Sx-Tw||^p \le \varphi([\{2a||Ax-Aw||^{2p}\}/\{||Sx-Sw||^p+||Ax-Aw||^p\}]\lozenge[b\,max\{||Sx-Sw||^p, ||Aw-Ax||^p\}]\lozenge[c.0])$$
$$< [2a||Aw-Ax||^p]\lozenge[b||Sx-Sw||^p]\lozenge 0$$
$$< ||Sx-Sw||^p\,\alpha\,max\{2a, b, 0\}$$
$$< ||Sx-Sw||^p,$$

a contradiction. Thus (2.10) holds. Since A is continuous at w, (2.10) implies that S is continuous at w. Similarly, if B is continuous at w then T is continuous at w. The uniqueness and continuity of mappings S and T can be proved easily.

This completes the proof.

Replacing ◊ by +, i.e, a◊b=a+b for all a, b ε R^+, **Theorem 2.1** reduces to the following Corollary:

Corollary 2.5. *Let A, B, S and T be four self-mappings of a normed space X and let C be a closed and convex subset of X satisfying the following condition:*

$$||Sx-Ty|^p \le \varphi\,\big([\,\{2a||Ax-By|^{2p}\}/\{||Sx-By||^p+||Ty-Ax||^p\}\,]+[b\,\max\{||Sx-By||^p,\ ||Ty-Ax||^p\}]+[c\,\min\{||Sx-Ax||^p,\ ||Ty-By||^p\}]\big) \tag{2.11}$$

for all x, y ε C for which $||Sx-By||^p+||Ty-Ax||^p \neq 0$, where $0<a<½$, $0<b<1$, $p>0$, $c\ge0$, $0<2a+b+c<½$; and $\varphi:R^+\to R^+$ is an u. s. c. function such that $\varphi(t)<t$ for each $t>0$. Suppose that the set inclusion relation (2.2) and the inductive sequence relation (2.3) satisfy. If A and B are continuous at z, and if (A, S) is weakly A-biased and (B, T) is weakly B-biased, then A, B, S and T have a unique common fixed point at w=Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Remark 1. In Theorem 2.5 of Ciri´c and Um`e [2], the argument of a function φ(t) is

$$t = ([2a||Ax-By||^{2p}]/\,[||Sx-By||^p+||Ty-Ax||^p]) + (1-a)\,max\,\{||Sx-By||^p,\ ||Ty-Ax||^p\} + c\,min\{||Sx-Ax||^p,\ ||Ty-By||^p\}$$

It is easy to verify that **Theorem 2.4** remains true with this argument of φ (t).

Remark 2. In Theorem 2.6 of Shahzad and Sahar [16], the argument of a function φ(t) is

$$t = ([a||Ax-By||^{2p}]/[max\{||Sx-By||^p,||Ty-Ax||^p\}])+min\{||Sx-By||^p,\ ||Ty-Ax||^p\},$$

and coefficient c is zero. It is easy to verify that **Theorem 2.4** remains true with this argument of φ(t) and c>0.

Replacing ◊ by '*max*', that is, a ◊b=*max* {a, b} for all a, b ε R^+, in the inequality (2.6), **Theorem 2.4** reduces to the following Corollary:

Corollary 2.6. *Let A, B, S and T be four self-mappings of a normed space X and let C be a closed and convex subset of X satisfying the following condition:*

$$||Sx-Ty||^p \le \varphi(max\{2a||Ax-By||^{2p}/[||Sx-By||^p+||Ty-Ax||^p],\ b\,max[||Sx-By||^p,\ ||Ty-Ax||^p],c\,min[||Sx-Ax||^p,\ ||Ty-By||^p]\}$$

for all x, y ε C for which $||Sx-By||^p+||Ty-Ax||^p \neq 0$, where $0<a<1/2$, $0<b<1$, $p>0$, $c\ge0$, $max\{2a, b, c\}<1$; and $\varphi:R^+\to R^+$ is an u.s.c. function such that $\varphi(t)<t$ for each $t>0$. Suppose that the set-inclusion relation (2.2) and the inductive sequence relation (2.3) satisfy. If A and B are continuous at z, and if (A, S) is weakly A-biased and (B, T) is weakly B-biased, then A, B, S and T have a unique common fixed point at w=Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Part- II: Fixed point theorem for (owc) maps

In this section, we will use the (owc) mappings which generalizes Theorem 1 of Pathak and Verma [13]. First we give our main Theorem of this section.

Theorem 2.7. *Let A, B, S and T be four self-mappings of a normed space X and let C be a closed and convex subset of X satisfying the following condition:*

$$\left\|Sx-Ty\right\|^p \le \frac{\left(a\left\|Ax-By\right\|^{2p}\right)\langle\rangle\left(b\max\{\left\|Ax-Sx\right\|^{2p},\ \left\|By-Ty\right\|^{2p}\}\right)}{max\left\{\left\|By-Sx\right\|^p,\ \left\|Ax-Ty\right\|^p\right\}} \tag{2.12}$$

for all x, y ε C for which $max\{||Sx-By||^p,\ ||Ty-Ax||^p\} \neq 0$, where $0<a<1$, $0<b<1$, $p\ge0$ and ◊ satisfies property-α with $0<a\alpha<1$, $0<b\alpha<1$. Suppose that the set-inclusion relation (2.2) and the inductive sequence relation (2.3) satisfy.

If A and B are continuous at z, and if (A, S) is (owc) and (B, T) is weakly compatible or vise-versa, then A, B, S and T have a unique common fixed point at w=Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Proof. As in Theorem 1 of Pathak and Verma [13] and as in previous Theorem 2.1, we can prove that z is a coincidence point of A, B, S and T. That is,

$$Az = Bz = Sz = Tz = w. \tag{2.13}$$

Now, suppose that (A, S) is (owc) at some point $\xi \in C(A, S)$, the set of coincidence point of A and S, then by definition, $A\xi=S\xi=\eta$ (say) and $AS\xi=A\eta=SA\xi=S\eta$. Since (B, T) is weakly compatible at z, we have BTz=TBz, i.e., $B\eta=T\eta$. If $A\eta\neq B\eta$, then from condition (2.12), we have

$$\|A\eta - B\eta\|^p = \|S\eta - T\eta\|^p \le \frac{\left(a\|A\eta - B\eta\|^{2p}\right)\Diamond\left(b\,max\{\|A\eta - S\eta\|^{2p},\ \|B\eta - T\eta\|^{2p}\}\right)}{max\{\|B\eta - S\eta\|^p, \|A\eta - T\eta\|^p}$$

$$\le a\alpha\|A\eta - B\eta\|^p < \|A\eta - B\eta\|^p,$$

a contradiction. Thus $A\eta=B\eta$. If $T\eta\neq\eta$, then from (2.12), we have

$$\|S\xi - T\eta\|^p \le \frac{\left(a\|A\xi - B\eta\|^{2p}\right)\Diamond\left(b\,max\{\|A\xi - S\xi\|^{2p},\ \|B\eta - T\eta\|^{2p}\}\right)}{\left[max\{\|B\eta - S\xi\|^p,\ \|A\xi - T\eta\|^p\}\right]}$$

$$\le a\,\alpha\|A\xi - B\eta\|^p < \|A\xi - B\eta\|^p = \|\eta - T\eta\|^p,$$

a contradiction. Thus, $T\eta=\eta$. Hence η is a common fixed point of A, B, S and T. The alternative case can be verified similarly. The uniqueness of η is easy to prove. The continuity of S and T, whenever A and B are continuous, can be shown in the similar way of Theorem 1 [13], as ◊ is continuous. This completes the proof.

Remark 3. The replacement of (owc) of a pair by the weakly S-biased map (or weakly S-biased map) of that pair, do not ensure the existence of common fixed point. For, suppose that the pair (A, S) is weakly A-biased instead of (owc) of (A, S), then by definition, we have at $z \in C$, $\|ASz-Az\| \le \|SAz-Sz\|$, i.e., from (2.12),

$$\|Aw-w\| \le \|Sw - w\|, \tag{2.14}$$

Let us show that Sw=w, and hence Aw=w. From (2.12) we have

$$\|Sw - w\|^p = \|Sw - Tz\|^p \le \frac{\left(a\|Aw - Bz\|^{2p}\right)\Diamond\left(b\max\{\|Aw - Sw\|^{2p}, \|Bz - Tz\|^{2p}\}\right)}{\max\left\{\|Bz - Sw\|^p, \|Aw - Tz\|^p\right\}}$$

$$\le \alpha\ max\left\{a\|Aw - w\|^{2p},\ b\ \|Aw - Sw\|^{2p}\right\} / \|w - Sw\|^p.$$

If $a\|Aw-w\|^{2p}$ is '*max*' then it yields $\|Sw-w\|^p \le a\alpha\|Aw-w\|^{2p}/\|w-Sw\|^p$, that is,

$$\|Sw-w\|^{2p} < a\alpha\|w-Sw\|^{2p} \tag{2.15}$$

a contradiction. Thus '*max*'= $b\|Aw-Sw\|^{2p}$, and so that

$$\|Sw-w\|^{2p} \le b\alpha\|Aw-Sw\|^{2p}. \tag{2.16}$$

This inequality is important. It indicates, as well as forces the mappings A and S to have a common fixed point w iff Aw=Sw, that is, the pair (A,S) is (owc). Thus the replacement of condition of weak compatibility of one of the mapping-pairs to (owc) is possible, but weakly A-biased is not possible. Similar argument can be stated for weakly S-biased. This argument also establishes the Remark 2 of [13] that the weak compatibility of one of the pairs is necessary. This remark especially underlines as well as differs the notions of (owc) and weakly-biased maps, as mentioned in the introduction part.

Replacing ◊ by +, then the **Theorem 2.7** reduces to the following Corollary:

Corollary 2.8. *Let A, B, S and T be four self-mappings of a normed space X and let C be a closed and convex subset of X satisfying the following condition:*

$$\|Sx-Ty\|^p \le \frac{\left(a\|Ax-By\|^{2p}\right)+\left(b\max\{\|Ax-Sx\|^{2p},\ \|By-Ty\|^{2p}\}\right)}{max\left\{\|By-Sx\|^p,\ \|Ax-Ty\|^p\right\}} \tag{2.17}$$

for all x, y ε C for which max{||Sx−By||^p, ||Ty−Ax||^p} ≠ 0, where 0<a<½, 0<b<½, p≥0 and 0<a+b<1. Suppose that the set-inclusion relation (2.2) and the inductive sequence relation (2.3) satisfy. If A and B are continuous at z, and if (A, S) is (owc) and (B, T) is weakly compatible or vise-versa, then A, B, S and T have a unique common fixed point at w=Tz. Further, if A and B are continuous at w, then S and T are continuous at w.

Remark 4. We have shown that the condition of weakly compatibility (and (owc), as well) imply to weakly S-biased maps and weakly A-biased maps, but not conversely. Therefore, keeping this fact in view, we have not replaced the weakly biased maps of Theorem 2.1 and Theorem 2.5 of Ciri´c and Um´e in [2], by (owc) condition (as, it will be a reverse process). Further, since weakly compatibility necessarily imply to (owc), we have replaced in Theorem 1 [13], the condition of weakly compatibility of one of the mapping-pairs to (owc) condition. Further note that, we have not replaced the condition of weakly compatibility of Theorem 1 ([13]) directly to weakly S-biased map (or, weakly A-biased map). Thus, in order to generalize Theorem 1 [13], we have replaced only the weakly compatibility condition of one of the mapping pairs to (owc), but not by weakly S-biased map (or, weakly A-biased map) and the other pair to weakly-biased maps. This facts are the actual difference between our Theorems; in which the results of first section uses only weakly-biased maps, and the result of second section is restricted to (owc) of one pair.

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