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# GRAPH THEORETICAL REPRESENTATION OF KNOT SYMMETRIC ALGEBRA

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*ABSTRACT*

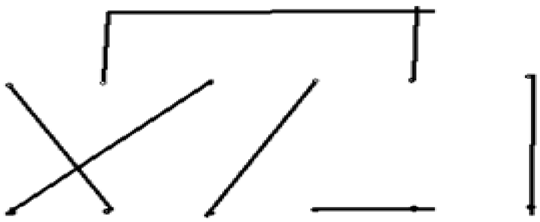
*In this paper we introduce Graph theoretical representation of Knot symmetric Algebra.*

## INTRODUCTION

In [Br], Brauer algebra was introduced by Richard Brauer (1937) in connection with the finding irreducible representation of the orthogonal group. Generators of Brauer algebra were represented by a graph with 2n vertices arranged in two rows such that each row contains n vertices. In [KM], we introduced a new class of algebras which are known as Knot symmetric algebras. Brauerdiagram (graph) motivated as to represent every generator of Knot symmetric algebras as a special type of graph which we call them as Knot graphs.

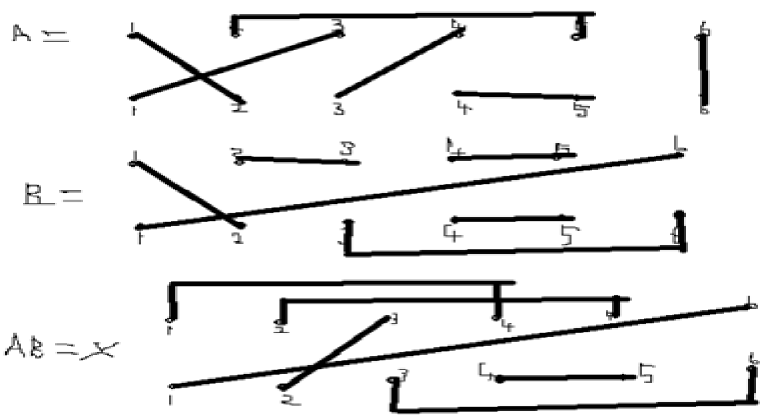
## 1. PRELIMINARIES

**Brauer algebras 1.1.** For $k \in Z$ and $x \in C$, the Brauer algebra $B_k(x)$ is the algebra over C whose basis consists of all diagrams on 2k vertices that have any combination of horizontal and vertical edges. An example of Brauer diagram is in below Fig 1



The dimension formula for $B_k(x)$ is (2k-1)!

Where (2k-1)! = (2k-1)(2k-3)…..3 1 . Multiplying Brauer diagrams introduce a parameter x which comes in to play when a loop forms in the middle rows of two diagram being multiple .A loop can be formed by two or more horizontal edges in the middle rows. When this occurs the loops disappear and we multiply the resulting diagram by $x^l$ where l is the number of loops in the middle rows. For example n=6 Note that horizontal and vertical edges can appear in the product of two diagrams via a sequences of edges that starts and ends with a vertical edge and which may have horizontal edges in the middle.



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**Knot symmetric algebras 1.2**
Let $S_n$ denote the symmetric group of order n. Every element of $S_n$ can be represented as Brauer diagram (graph) with 2n vertices and with out horizontal edges [Br]. Let $\pi \in S_n$ the vertices of π are represented in two rows such that each row contains n vertices. The vertices of each row is indexed with 1, 2….., n from left to right in order. Let E(π) denote the set of all edges of π.

(i.e) $E(\pi)=\{e_i=(i, \pi(i));\ 1<i<n\}$

Define S(π) is a subset of $E(\pi)\times E(\pi)$ such that $S(\pi)=\{(e_i, e_j),\ i<j\}$. It is obvious that $\left|S(\pi)\right| = \frac{n(n-1)}{2}$

**Example. 1** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \in S_4$ is represented by fig 1

For the Fig 1

$E(\pi)=\{e_1=(1,3),\ e_2=(2,4),\ e_3=(3,2),\ e_4=(4,1)\}$

$S(\pi)=\{(e_1,e_2),\ (e_1,e_3),\ (e_1,e_4),\ (e_2,e_3),\ (e_2,e_4),\ (e_3,e_4)\}$

Let $f_\pi$ be a mapping from S(π) to {-1,0,1} such that

$$f_\pi(e_i,e_j)=\begin{cases} 0 & if\ \pi(i) < \pi(j0 \\ 1\ or -1 if \pi(i) > \pi(j) \end{cases}$$

and $f_\pi(e_i,e_j) + f_\pi(e_j,e_i) = 0$

Fig: 1

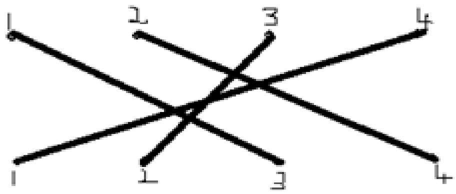
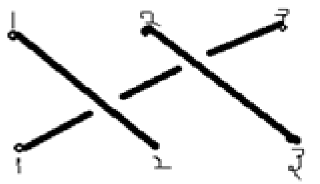


Fig: 2



**Knot mapping 1.3.** A mapping $f_\pi$ defined above is called a Knot mapping. Refer the above Fig 2. We have

$e_1 = (1, 2),\ e_2= (2, 3),\ e_3= (3, 1)$

$f_\pi(e_1,e_2)=0$ , $f_\pi(e_1,e_3)=1$, $f_\pi(e_3,e_1) = -1$

**Knot Number 1.4** Define $K(\pi)=\{(e_i, e_j) \epsilon S(\pi);\ \pi(i) > \pi(j)\}$

**Definition 1.5** $\left|K(\pi)\right|$ is called Knot number of π .Let x be indeterminate. Define $N(\pi)=\{x^m f_\pi\ ;\ m \epsilon\ z,\ f_\pi$ is a Knot mapping}

For any two Knot mapping $f_\pi$ and $g_\pi$ .

Define $E(f_\pi, g_\pi) = \{(e_i, e_j) \epsilon K(\pi) : f_\pi(e_i, e_j) + g_\pi(e_i, e_j)=0\}$

**Knot Relation 1.6** Define a relation ~ in N(π) such that $x^m f_\pi \sim x^l g_\pi$ if
(i) m = l and $f_\pi = g_\pi$ or
(ii) $l\text{-}m = 2 \sum_{(e_i,e_j)\in E(f_\pi,g_\pi)} f_\pi(e_i, e_j)$ This relation is called Knot relation

**Knot multiplication 1.7** Let $\overline{N(\pi)} = N(\pi)/\sim$ That is $\overline{N(\pi)}$ is the collection of disjoint equivalence classes with respect to the Knot relation. Define $T_n = \left\{(\pi, x^m f_\pi) : \pi \in S_n,\ f_\pi \in \overline{N(\pi)} \quad \textit{and } m \textit{ is an integer}\right\}$

We define multiplication in $T_n$ as follows:

Let a, b $\in T_n$ and $a=(\pi, x^m f_\pi)$, $b=(\sigma, x^l g_\pi)$.

Define $ab = (\sigma \, o \, \pi, x^{m+l+\alpha} h_{\sigma o \pi})$ where $\alpha$ and $h_{\sigma o \pi}$ are defined as follows:

Let $(e_i', e_j') \in S(\sigma o \pi)$, $(u_i, u_j) \in S(\pi)$, $(v_p, v_q) \in S(\sigma)$, $p, q \in \{\pi(i), \pi(j)\}$, $f_\pi(u_i, u_j) = u$ and $g_\pi(v_p, v_q) = v$.

Now $\alpha = \sum_{(e, e_j'{}_i) \in s(\sigma o \pi)} \alpha(e_i', e_j')$

where $\alpha(e_i', e_j') = (u+v)|uv|)$ and $h_{\sigma o \pi}(e_i', e_j') = (u+v)(1-\delta_{u,v})$ where $\delta(u,v) = \begin{cases} 0 & if \quad u \neq v \\ 1 & if \quad u = v \end{cases}$

**Theorem 1.8.** The Knot multiplication is associative in $T_n$.

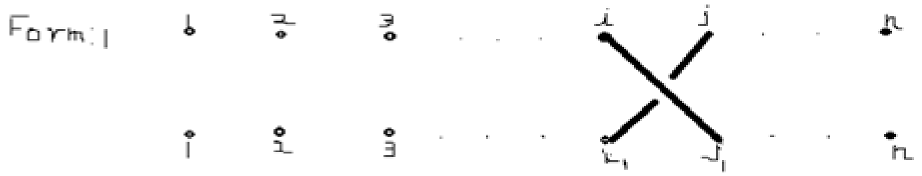
**Theorem 1.9.** $FT_n$ is an algebra

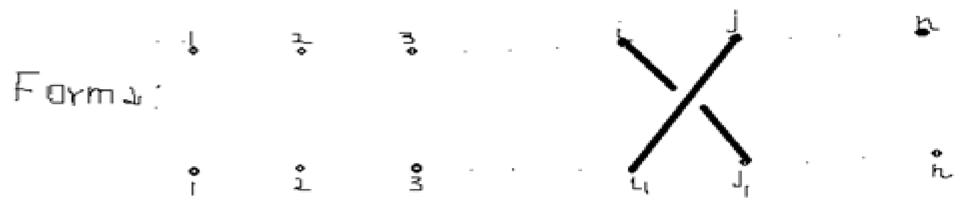
This algebra is called as Knot symmetric algebra

## 2. KNOT GRAPHS

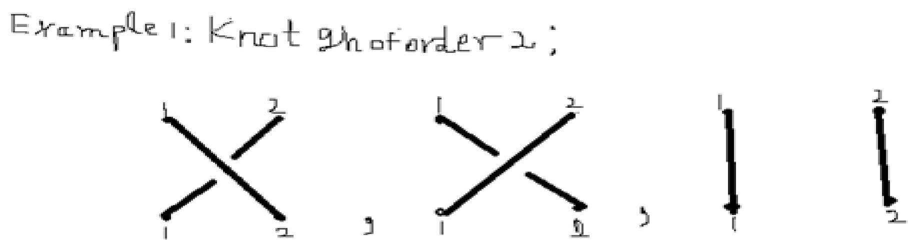
Let $S_n$ be the symmetric group of order n and $\pi \in S_n$. A knot graph of order n is a special graph which is defined from $\pi$ as follows.

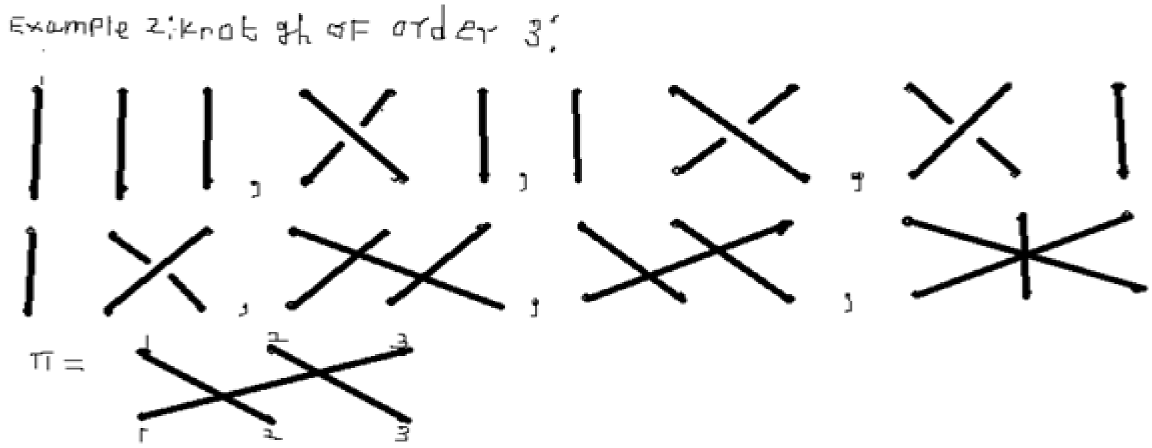
**Definition 2.1** we start with an element $\pi \in S_n$, $\pi$ can be represented by a graph .Consider two edges $(i, \pi(i))$ and $(j, \pi(j))$ where vertices i and j are in the upper row and $i_1$ and $j_1$ are in the lower row. If $i<j$, $i_1<j_1$ then edges are as in the Brauer diagram. If $i<j$ and $j_1<i_1$, then we draw edges in two forms as shown below.



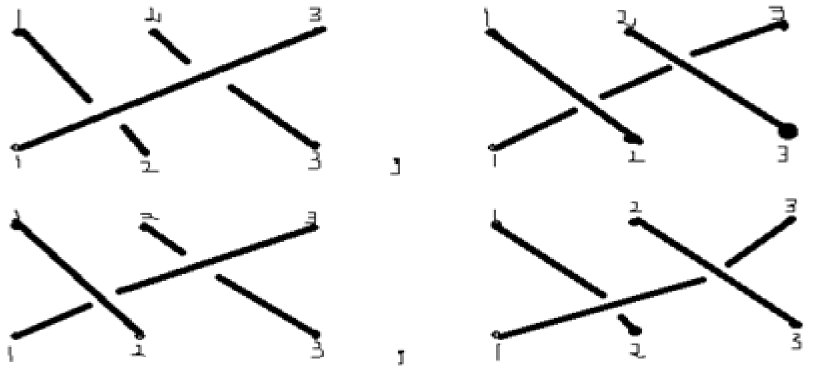


In form 1, we say $(i, i_1)$ is the upper edge than $(j, j_1)$. In this case we may also say that $(j, j_1)$ is lower than $(i, i_1)$. In form 2, we say that $(j,j_1)$ is the upper edge than $(i, i_1)$. In this case we may also say that $(i, i_1)$ is lower than $(j, j_1)$. The above graph is called Knot graph of order n with respect to $\pi$.






$K(\pi)=\{(e_1,e_3),(e_2,e_3)\}$ and $|K(\pi)|=2$ Hence number of Knot mappings of $\pi$ is $2^{|K(\pi)|} = 2^2 = 4$ The four Knot mappings of $\pi$ is described below:



**Definition 2.2.** $\pi \in S_n$ , $i<j$, $\pi(i) > \pi(j)$ we say there is a crossing and the edges are $(i, \pi(i))$ and $(j, \pi(j))$

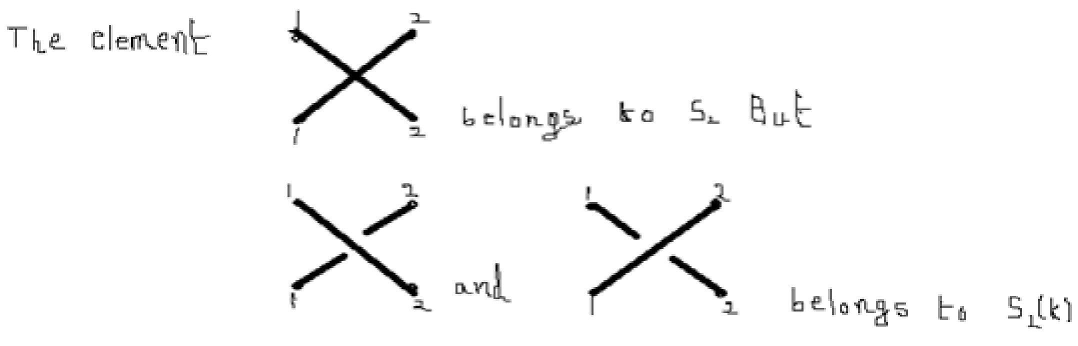
**Remark 2.3.** Given $\pi \in S_n$ there are $2^{|k(\pi)|}$ Knot graphs which is equal to $\{G_i(\pi)\}$ where i=1 to are $2^{|k(\pi)|}$

**Definition 2.4.** If $G_i(\pi)$ is a Knot graph with respect to $\pi$, then $\pi$ is called underlying graph of $G_i(\pi)$

**Notation 2.5.** $S_n(k)$ denote the collection of all Knot graphs

**Remark 2.6.** $S_1(k) = S_1$.

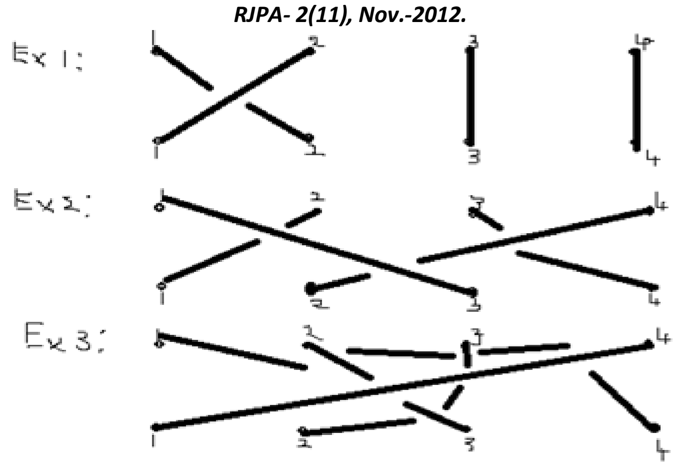
**Remark 2.7.** $S_2(k) \neq S_2$ , For




**Remark 2.8.** $S_n \neq S_n(k)$ $\forall$ $n \geq 2$

**Notation2.9.** A Knot graph with $\pi$ is denoted by $G_i(\pi)$, i=1,2….. $2^{K(\pi)}$ Examples of knot graphs:

When n = 4

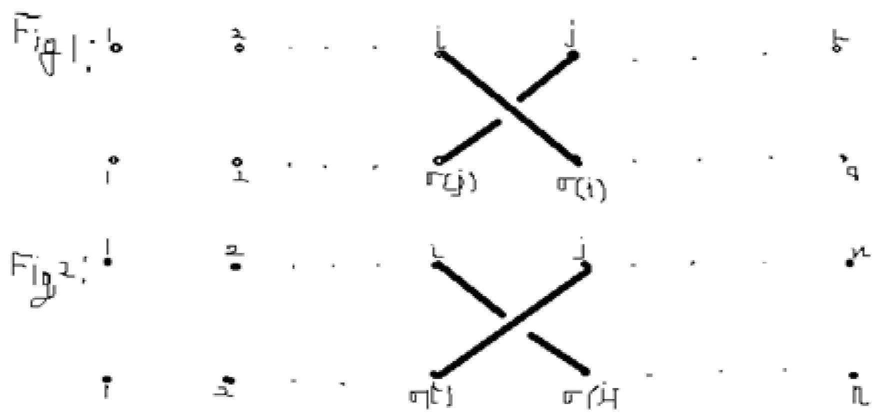


**Notation2.10**: we denote a Knot graph of order n as $G_n(\pi)$

**Theorem 2.11.** Every generator of Knot symmetric algebra FTn can be represented by a unique Knot graph of order n and every Knot graph of order n can be represented by a generator of Knot symmetric algebra $FT_n$ .

That is there is a one to one correspondence between the generators of Knot symmetric algebra $FT_n$ and the Knot Graph of order n

**Proof:** let $(\pi, f_\pi)$ be an generator of FTn. Now $\pi \in S_n$ the vertices of $\pi$ are represented in two rows such that each row contains n vertices.Let$e_i=(i ,\pi(i ))$ and $e_j=(j ,\pi(j))$ be two edges. If i <j and $\sigma(i)>\sigma(j)$ we draw in such a way that $e_i$ is upper than $e_j$ if $f_\pi(e_i,e_j)=1$ The diagram is refer in below **Fig 1:** Next we refer the below **Fig 2** as follows: we draw in such a way that $e_j$ is lower than $e_i$ with respect to $f_\pi$ if $f_\pi(e_i, e_j) = -1$. The diagram of Fig 1 and Fig 2 is drawn as follows:



Thus we get a Knot graph of order n corresponding to $(\pi,f_\pi)$. Now we will prove that every Knot graph of order n represent a generator.

Let $G_n$ be a Knot graph of order n .Now $\pi(G_n) \in S_n$ we denote $\pi$ instead of $\pi(S_n)$. Define $f_\pi$: $S(\pi) \rightarrow \{-1,0,1\}$ as

$$f_\pi(e_i,e_j)=\begin{cases} 0 & if\, \pi(i) < \pi(j) \\ 1 & if\ e_i\ is\, upper\, than\, e_j \\ -1 & if\ e_i\ is\, lower\, than\, e_j \end{cases}$$

It is obvious that the graph represented by $(\pi, f_\pi)$ is $G_n$

**Example.** when n=4,

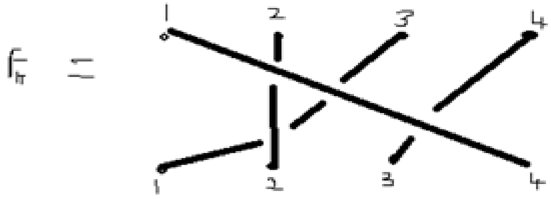
let $\pi=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ and $E(\pi)=\{e_1=(1,4),e_2=(1,2),e_3=(3,1),e_4=(4,3)\}$

$S(\pi)=\{(e_1,e_2),(e_1,e_3),(e_1,e_4),(e_2,e_3),(e_2,e_4),(e_3,e_4)\}$

Let $f_\pi$ be defined as follows:

$f_\pi(e_1,e_2) = 1, f_\pi(e_1,e_3) = 1, f_\pi(e_1,e_4) = 1, f_\pi(e_2,e_3) = 1, f_\pi(e_2,e_4) = 0, f_\pi(e_3,e_4) = 0.$

Now the Knot Graph represented by $(\pi, f_\pi)$ is shown below: In this example the edge $e_1$ is upper than $e_2$ with respect to $f_\pi$



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