Research Journal of Pure Algebra -3(1), 2013, Page: 1-7
Available online through www.rjpa.info ISSN 2248–9037

FIXED POINT THEOREMS IN DISLOCATED QUASI METRIC SPACES USING THE NOTION OF A- CONTRACTIONS

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(Received on: 15-11-12; Revised & Accepted on: 08-12-12)

ABSTRACT

A fixed point theorem for a self map T (not necessarily continuous) on a complete dq- metric space is proved. Incidentally, we show that a result of Rajesh. S., Ansari.Z.K. and Manish Sharma [12] in its revised from (Theorem 2.1) is not valid through an example . We also extend a fixed point theorem for two self maps on a dislocated metric space of Rajesh.et.al [12] to dislocated quasi metric spaces.

Mathematical Subject classification: *47 H 10, 54 H 25, 55M 20.*

Key words: *Dislocated metric space, dq-metric space, contractive condition, fixed point, A – contraction.*

0. INTRODUCTION

The concept of dislocated metrics was studied under the name of metric domains in the context of domain theory in [2]. As a generalization of metrics where the self distance for any point need not to be equal to zero, Hitzler [6] and Hitzler and Seda [7], introduced the notion of dislocated metric spaces and generalized the celebrated Banach contraction principle in such spaces. These metrics play a very important role not only in topology but also in other branches of Science involving mathematics especially in logic programming and electronic engineering.

Zeyada.et.al [15] initiated the concept of dislocated quasi metric space and generalized the result of Hitzler and Seda [7] in dislocated quasi metric spaces. Recently, the study in such spaces is followed by Isufati [8] and C. T. Aage and J. N. Salunke [1]. In [8], the author proved some fixed point results in dislocated quasi metric spaces involving continuous contraction mappings which exist in the literature of usual complete metric space. In [1] the authors proved Kannan's [10] fixed point theorem and Lj. B. Ciric's [5] generalized contraction on complete metric spaces in the setting of dislocated quasi metric spaces. On the other hand, it is well known that Banach contraction principle is a pivotal result in the metric fixed point theory. This principle has been generalized by various authors by putting different types of contractive conditions. In the sequel, D.S. Jaggi [9] proved a fixed point theorem using rational type of contractive condition which generalized the Banach contraction principle in complete metric spaces.

In 2008 Akram.et.al [3] introduced a larger class of contractions, called A- contractions and showed that the class of A- contractions is a proper super class of Kannan's [10], Khan's [11], Bianchini's [4] and S. Reich [13], type contractions. Also, the authors proved fixed point theorems for A-contraction in complete metric spaces. Rajesh Srivastava.et.al [12] presented a version of D.S. Jaggi's [9] fixed point theorem in the context of dislocated quasi metric spaces. Further a common fixed point theorem is also obtained in complete dislocated metric space, using the notion of A-contraction in Rajesh.et.al [12].

In this paper, we observe that Theorem 3.1 of Rajesh.et.al [12] is not meaningful and provide a revised statement to make this theorem meaningful. Further we prove a fixed point theorem for a self map on a complete dislocated quasi metric space, satisfying a condition similar to the one in Rajesh.*et.al* [12]. We also extend a fixed point theorem for a A- contraction pair on a dislocated metric space (Theorem 3.7 of Rajesh.*et.al* [12]) to dislocated quasi metric spaces.

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1. PRELIMINARIES

Definition 1.1: [14] Sastry. K.P.R., Vali. S.K., SrinivasaRao. Ch. and Rahamatulla. M.A. [14] Let X be a non-empty set and let d: $X \times X \rightarrow [0,\infty)$ be a function, called a distance function, satisfying one or more of $d_1 - d_5$.

d_1: $d(x,x) = 0 \; \forall \, x \in X$,
d_2: $d(x,y) = d(y,x) = 0 \implies x = y \; \forall \, x, y \in X$,
d_3: $d(x,y) = d(y,x) \; \forall \, x, y \in X$,
d_4: $d(x,y) \leq d(x,z) + d(z,y) \; \forall \, x, y, z \in X$,
d_5: $d(x,y) \leq \max\{d(x,z), d(z,y)\} \; \forall \, x, y, z \in X$.

(i) If d satisfies d_2, d_3 and d_4 then d is a called a dislocated metric and (X,d) is called a dislocated metric space.
(ii) If d satisfies d_1, d_2 and d_4 then d is called a quasi metric and (X,d) is called a quasi metric space.
(iii) If d satisfies d_2 and d_4 then d is called a dislocated quasi metric (or)dq-metric and (X,d) is called a dq-metric space.
(iv) If d satisfies d_1, d_2, d_3 and d_4 then d is called a metric and (X,d) is called a metric space.

Definition 1.2: A sequence $\{x_n\}$ in a dq–metric space (X,d) is called Cauchy if to $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0$, $d(x_m, x_n) < \epsilon$.

Definition 1.3: Zeyada.F.M., Hassan.G.H. and Ahmed.M.A. [15] A sequence $\{x_n\}$ in a dislocated quasi metric space(X,d) is said to be dislocated quasi converges (or) dq - converges to x, if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In this case x is called a dq – limit of $\{x_n\}$ and we write $x_n \rightarrow x$.

Proposition 1.4: Zeyada.F.M., Hassan.G.H. and Ahmed.M.A. [15] Every convergent sequence is a dq- metric space is Cauchy.

Definition 1.5: Zeyada.F.M., Hassan.G.H. and Ahmed.M.A. [15] A dq – metric space (X,d) is complete, if every Cauchy sequence in it is dq – convergent.

Lemma 1.6: Zeyada.F.M., Hassan.G.H. and Ahmed.M.A. [15] Every subsequence of a dq – convergent sequence to a point x_0 is dq – convergent to x_0.

Definition 1.7: Let $(X,d), (Y,d')$ be dq - metric spaces. Suppose $: X \rightarrow Y$, we say that f is continuous, if

$$\{x_n\} \rightarrow x \Rightarrow fx_n \longrightarrow fx \;\; in \;\; Y .$$

Definition 1.8: Zeyada.F.M., Hassan.G.H. and Ahmed.M.A. [15] Let (X,d) be a dq- metric space. A mapping $f : X \rightarrow X$ is called a contraction if there exists $0 \leq \lambda < 1$ such that

$$d(f(x), f(y)) \leq \lambda d(x,y) \text{ for all } x, y \in X.$$

Lemma 1.9: eyada.F.M. Hassan.G.H. and Ahmed.M.A. [15] Let (X,d) be a dq- metric space. If $f : X \rightarrow X$ is a contraction function, then $f^n(x_0)$ is a Cauchy sequence for each $x_0 \in X$.

Lemma 1.10: Zeyada.F.M., Hassan.G.H. and Ahmed.M.A. [15] dq – limits in a dq – metric space are unique.

Theorem 1.11: Zeyada.F.M., Hassan.G.H. and Ahmed.M.A. [15] Let (X,d) be a complete dq - metric space and let $f : X \rightarrow X$ be a continuous contraction function. Then f has a unique fixed point.

RAJESH.S. ANSARI.Z.K. AND MANISH SHARMA [12] CLAIMED THE FOLLOWING RESULT

Theorem 1.12 (Rajesh's., Ansari.Z.K. and Manish Sharma [12], Theorem 3.1) Let T be a continuous self map defined on a complete metric space (X,d). Further let T satisfy the following contraction condition

$$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \qquad (1.12.1)$$

for all $x, y \in X, x \neq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$, Then T has unique fixed point.

Note 1.13: Condition (1.12.1) is meaningless if $d(x,x) = 0$. Also we note that $x \neq y$ and $d(x,y) = 0$ may happen in a dq - metric space.

Hence a revised statement of Theorem (1.12) to make this meaningful is given in Theorem (2.1).

Definition 1.14: Akram.M. Zafar.A.A. and Siddiqui.A.A. [3] A self map T on a metric space (X,d) satisfying

$$d(Tx,Ty) \leq \alpha \max \begin{Bmatrix} d(x,Tx) + d(y,Ty), \\ d(y,Ty) + d(x,y), \\ d(x,Tx) + d(x,y) \end{Bmatrix} \tag{1.14.1}$$

for all $x, y \in X$ *and some* $\alpha, \beta \in [0, \frac{1}{2})$ is called a A- contraction.

Note 1.15: The notion of A-contraction can be extended in a natural way to dq – metric spaces also with metric space replaced by dq – metric space. We use this notion of A - contraction in dq – metric space in the next section.

2 . MAIN RESULTS

WE REVISE THE STATEMENT OF THEOREM (1.12) AS FOLLOWS, TO MAKE IT MEANINGFUL.

Theorem 2.1: Let T be a continuous self mapping defined on a complete dq-metric space (X,d). Further let T satisfy the following contractive condition

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \tag{2.1.1}$$

for all $x, y \in X$, *whenever* $d(x,y) \neq 0$.

Then T has a unique fixed point.

WE OBSERVE THAT THEOREM 2.1 IS NOT VALID IN VIEW OF THE FOLLOWING EXAMPLE.

Example 2.2: Let $X = \{0,1\}$, *define* $d(0,0) = 0$, $d(0,1) = 1$, $d(1,0) = 0$, $d(1,1) = 0$, and define $T : X \to X$ as $T0 = 1$ *and* $T1 = 0$. Then (X,d) *is* a complete dq- metric space, T satisfies (2.1.1) and T has no fixed Point.

NOW WE PROVE A THEOREM, SIMILAR TO THEOREM 2.1 WITHOUT ASSUMING CONTINUITY OF *T* WHICH HOLDS GOOD IN A COMPLETE DQ – METRIC SPACE.

Theorem 2.3: Let T be a self map defined on a complete dq-metric space (X,d), let T satisfy the condition

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{\max\{d(x,y), d(y,x)\}} + \beta d(x,y) \tag{2.3.1}$$

for all $x, y \in X$, *whenever* $d(x,y) \neq 0$ *(or)* $d(y,x) \neq 0$.

That is $d(x,y) + d(y,x) \neq 0$, *where* α, β *are non – negitive and* $\alpha + \beta < 1$. Then T has a unique fixed point.

Proof: Suppose

$\max\{d(x,Tx), d(Tx,x)\} \neq 0$. Put $y = Tx$ in (2.3.1). We get

$$d(Tx,TTx) \leq \alpha \frac{d(x,Tx)d(Tx,TTx)}{\max\{d(x,Tx), d(Tx,x)\}} + \beta d(x,Tx)$$

$$\leq \alpha d(Tx,TTx) + \beta d(x,Tx)$$

$$\therefore d(Tx,TTx) \leq \frac{\beta}{1-\alpha} d(x,Tx) \leq \frac{\beta}{1-\alpha} \max\{d(x,Tx), d(Tx,x)\}$$

Similarly we can show that (by replacing x with Tx and y *with* x in (2.3.1))

$$d(TTx,Tx) \leq \frac{\beta}{1-\alpha} \max\{d(x,Tx), d(Tx,x)\}$$

$\therefore max\,\{d(Tx, TTx\,), d(T\,Tx, Tx) \leq \frac{\beta}{1-\alpha}\max\{d(x, Tx), d(Tx\,, x)\}$ (2.3.2)

Let $x_0 \in X$. Define the sequence $\{x_n\}$ iteratively as follows:

$x_1 = Tx_0, x_2 = Tx_1, \ldots\ldots, x_{n+1} = Tx_n$.

Suppose, for some n, $x_{n+1} = x_n$. Then $Tx_n = x_{n+1} = x_n$, so that x_n is a fixed point of T.

Now suppose that $x_n \neq x_{n+1}\ for\ n = 0,1,2, \ldots.$ Then, clearly

$\max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\} \neq 0\,, for\ n = 1,2,3, \ldots.$

Hence, by (2.3.2) $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\} \leq \frac{\beta}{1-\alpha}\max\{d(x_{n+1}, x_n), d(x_n, x_{n+1})\}$

Since $\frac{\beta}{1-\alpha} < 1$, this shows that
$\leq \left(\frac{\beta}{1-\alpha}\right)^n \max\{d(x_0, x_1), d(x_1, x_0)\}$ and hence the sequence $\{x_n\}$ is a Cauchy sequence.

$\therefore x_n \to u\ for\ some\ u \in X$ (2.3.3)

Case (i): Suppose $\max\{d(x_n, u), d(u, x_n)\} \neq 0\ for\ large\ n$. Then, by (2.3.1),

$d(x_{n+1}, Tu) = d(Tx_n, Tu)$

$\leq \alpha \frac{d(x_n, Tx_n)d(u, Tu)}{\max\{d(x_n, u), d(u, x_n)\}} + \beta d(x_n, u), if\ x_n \neq u$ for large n .

$\to 0\ as\ n \to \infty.$

Similarly
$d(Tu\,, x_{n+1}) \to 0\ as\ n \to \infty.$

Hence $x_{n+1} \to Tu$

$But\ x_{n+1} \to u$ (by (2.3.3))

Consequently by lemma 1.10, $Tu = u$.

Case (ii): Suppose $x_n = u$ for infinitely many . Then for such $n, x_{n+1} = Tx_n = Tu$. Then

$d(x_n, x_{n+1}) < \epsilon\ and\ d(x_{n+1}, x_n) < \epsilon\ for\ n \geq N$. and hence $d(u\,, Tu) < \epsilon\ and\ d(Tu\,, u) < \epsilon$.

This being true for every $\epsilon > 0$ follows that

$d(u\,, Tu) = 0 = d(Tu\,, u)$

Hence $Tu = u$. Thus u is a fixed point of T.

UNIQUENESS
First suppose that $Tx = x$. Then $d(x, x) \neq 0$.
$\Rightarrow d(x, x) = d\,(Tx\,, Tx) \leq \alpha \frac{d(x, x)d(x, x)}{\max\{d(x, x), d(x, x)\}} + \beta d(x, x)$

$\leq \alpha\, d\,(x, x) + \beta\, d(x, x)$

$\leq (\alpha + \beta)d(x, x)$

$< d(x, x)$, a contradiction,

$\therefore d(x, x) = 0$, if x is a fixed point of T (2.3.4)

Suppose that $x\ and\ y\ in\ X$ are two distinct fixed point of T. That is $Tx = x\ and\ Ty = y$.

Then $\max\{d(x,y), d(y,x)\} \neq 0$ and hence by (2.3.1), we have

$$d(x,y) = d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{\max\{d(x,y),d(y,x)\}} + \beta d(x,y)$$

$$= \beta d(x,y) \quad \text{(by (2.3.4))}$$

$\therefore (1-\beta)d(x,y) \leq 0$

$\therefore d(x,y) = 0$, Similarly $d(y,x) = 0$.

$\therefore d(x,y) = d(y,x) = 0 \Longrightarrow x = y$.

Thus fixed point of T is unique. Thus the proof completes.

Corollary 2.4: Let T be a continuous self mapping defined on a complete dq – metric space (X,d). Further let T satisfy the contractive condition

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(y,x)} + \beta\, d(x,y) \qquad (2.4.1)$$

for all $x,y \in X, x \neq y\ and\ for\ some\ \alpha,\beta \in [0,1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point.

Proof: Since $\max\{d(x,y), d(y,x)\} \leq d(x,y) + d(y,x)$, it follows that (2.4.1) $\Longrightarrow$ (2.3.1)

Hence corollary 2.4 follows from Theorem 2.3. In continuiation of Note 1.15, we introduce the notion of A – contraction for two maps on a complete dislocated quasi metric space as follows:

Definition 2.5: Let (X,d) be a complete dislocated quasi metric space. Suppose $S,T : X \longrightarrow X$ are self maps satisfying $\max\{d(Sx,Ty), d(Ty,Sx)\} \leq \alpha\, max \begin{Bmatrix} d(x,Tx) + d(y,Ty), \\ d(y,Ty) + d(x,y), \\ d(x,Tx) + d(x,y) \end{Bmatrix}$ for all $x,y \in X\ and\ for\ some\ \alpha \in [0,\frac{1}{2})$. Then (S,T) is called A- contraction.

NOW WE EXTEND THE RESULT (THEOREM 3.7) OF [12] TO DISLOCATED QUASI METRIC SPACES AS FOLLOWS.

Theorem 2.6: Let (X,d) be a complete dislocated quasi metric space. Let $S,T : X \longrightarrow X$ be continuous self mapping. Suppose (S,T) is a A- contraction. Then $S\ and\ T$ have unique common fixed point.

Proof: Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_1 = Sx_0$, $x_2 = Tx_1$, ., $x_{2n} = Tx_{2n-1}$,…., $x_{2n+1} = Sx_{2n}$, ….. Now, for some $\alpha \in [0,\frac{1}{2})$,

$$d(Sx,TSx) \leq \alpha\, max \begin{Bmatrix} d(x,Sx) + d(Sx,TSx), \\ d(Sx,TSx) + d(x,Sx), \\ d(x,Sx) + d(x,Sx) \end{Bmatrix}$$

$$= \alpha\, max\{d(x,Sx) + d(Sx,TSx), 2d(x,Sx)\}$$

$$= \alpha\{d(x,Sx) + d(Sx,TSx)\}\ (or), 2\alpha d(x,Sx)$$

$\therefore d(Sx,TSx) \leq \frac{\alpha}{1-\alpha} d(x,Sx)\ (or)\ 2\alpha d(x,Sx)$

$\therefore d(Sx,TSx) \leq 2\alpha d(x,Sx)$

Now writing $x = x_{2n}$, we get

$d(Sx_{2n},TSx_{2n}) \leq 2\alpha d(x_{2n},Sx_{2n})$

i.e $d(x_{2n+1}, x_{2n+2}) \leq 2\alpha d(x_{2n}, x_{2n+1})$

$\therefore d(x_{2n+1}, x_{2n+2}) \leq Kd(x_{2n}, x_{2n+1})$, where $K = 2\alpha < 1$

Similarly, we can show that

$d(x_{2n}, x_{2n+1}) \leq Kd(x_{2n-1}, x_{2n})$

Hence $d(x_{n+1}, x_{n+2}) \leq Kd(x_{n-1}, x_n)\ for\ n = 1,2,3, \ldots.$

Consequently, we get

$d(x_{n+1}, x_{n+2}) \leq K^n d(x_0, x_1)$

Since $0 \leq K < 1, as\ n \to \infty, \quad K^n \to 0.$

Thus the sequence $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi metric space X. Therefore there exists a point $u \in X$ such that $x_n \to u$.

$\therefore$ The subsequence $\{Sx_{2n}\} \to u$ and $\{Tx_{2n+1}\} \to u$.

Since $S\ and\ T$ are continuous, so we have $Su = u\ and\ Tu = u.$

$\therefore u$ is a common fixed point of $S\ and\ T$.

UNIQUENESS

Let $u\ and\ v$ be common fixed points of $S\ and\ T$ Then by (1.14.1)

$$d(u,u) = d(Su,Tu) \leq \alpha\, max \begin{Bmatrix} d(u,u) + d(u,u), \\ d(u,u) + d(u,u), \\ d(u,u) + d(u,u) \end{Bmatrix}$$

$\therefore d(u,u) \leq 2\alpha d(u,u)$. Since $\alpha < \frac{1}{2}$, this shows that $d(u,u) = 0$,

Similarly, $d(v,v) = 0$.

$$\text{Now } d(u,v) = d(Su,Tv) \leq \alpha\, max \begin{Bmatrix} d(u,u) + d(v,v), \\ d(v,v) + d(u,v), \\ d(u,u) + d(u,v) \end{Bmatrix}$$

$$= \alpha d(u,v).\ \ So\ that\ d(u,v) = 0.$$

Similarly we have $d(v,u) = 0\ and\ so\ u = v$.

Hence $S\ and\ T$ have a unique common fixed point.

Thus the proof completes.

Corollary 2.7 (Theorem 3.7 of [12]): Let (X, d) be a complete dislocated metric space. Let $S, T : X \longrightarrow X$ be continuous mappings satisfying:

$$d(Sx,Ty)\} \leq \alpha\, max \begin{Bmatrix} d(x,Sx) + d(y,Ty), \\ d(y,Ty) + d(x,y), \\ d(x,Sx) + d(x,y) \end{Bmatrix} \text{ for all } x,y \in X\ and\ for\ some\ \alpha \in [0, \tfrac{1}{2}).$$

Then $S\ and\ T$ have common fixed point.

ACKNOWLEGEMENTS

The fourth author (M.A. Rahamatulla) is grateful to the authorities of Al-Aman College of Engineering and I.H. Faruqui Sir for granting permission to carry on this research.

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Source of support: Nil, Conflict of interest: None Declared