

DEGREE OF APPROXIMATION BY NÖRLUND SUMMABILITY MEANS OF LAGUERRE SERIES

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ABSTRACT

In this paper a general result on degree of Approximation of the Nörlund summability means of Fourier series with a weight has been obtained at the point $x = 0$ by Nörlund mean.

In 1981, S.N. Lal and K. N.Singh [4] obtained on the Absolute Summability of Fourier Series by Nörlund mean.

In this paper, we obtain the comparable result of [4] with degree of approximation by Nörlund summability of weighted Fourier series at the point $x=0$ by Nörlund mean.

Key word: Nörlund mean, Cesàro means, Harmonic means, Approximation of a Function.

INTRODUCTION:

Let $\sum u_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants and let us write

$$P(n) = P_n, \quad \text{where} \quad P_n = \sum_{k=0}^n p_k .$$

The sequence-to sequence transformation

$$t_n = \sum_{k=0}^n \frac{p_{n-k} S_k}{P_n} = \sum \frac{p_k S_{n-k}}{P_n}, \quad p_n \neq 0 \quad (1)$$

and the sequence $\{t_n\}$ of Nörlund mean of the sequence $\{S_n\}$ generated by the sequence of coefficient $\{p_n\}$.

The important particular cases of the Nörlund means are:

1. Harmonic means when $p_k = \frac{1}{k+1}$
2. Cesàro mean, when $p_k = \binom{k+\delta-1}{\delta-1}$, $\delta > 0$.
3. Nörlund mean (N, p_n) when $q_n = 1$ for all n .
4. Nörlund mean (N, q_n) when $p_n = 1$ for all n .

The Laguerre expansion of a function $f(x) \in L(0, \infty)$ is given by

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$$f(x) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \tag{2}$$

Where

$$a_n = \left\{ \Gamma(\alpha+1) \binom{n+\alpha}{n} \right\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_n^{(\alpha)}(y) dy \tag{3}$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre expansion of order $\alpha > -1$, defined by the function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) w^n = (1-w)^{-\alpha-1} \exp\left(\frac{-xw}{1-w}\right)$$

And existence of the integral (3) is presumed.

We write

$$\phi(y) = \{\Gamma(\alpha+1)\}^{-1} e^{-y} y^{\alpha} \{f(y) - f(0)\}$$

Gupta [2] estimated the order of the function by Cesàro means of the series (2) at the point $x = 0$ after replacing the continuity condition in Szego's theorem [7] by a much lighter condition.

Theorem: 1 If

$$F(t) = \int_0^t \frac{|f(y)|}{y} dy = o\left\{ \log \frac{1}{t} \right\}^{1+p}, t \rightarrow 0, -1 < p < \infty$$

and

$$\int_1^{\infty} e^{-y/2} y^{(3\alpha-3k-1)/3} |f(y)| dy < \infty$$

Then

$$\sigma_n^k(0) = o(\log n)^{p+1}$$

Provided $k > \alpha + \frac{1}{2}, \alpha > -1, \sigma_n^k(0)$ being the nth Cesàro mean of order k

Theorem: 2 For $k > \alpha > -1$

$$\sigma_n^k(f, 0) = o(n^{-1/4}) + o\left\{ \phi\left(\frac{1}{n}\right) \right\}$$

Provided

$$\int_0^t |df(y)| \leq A \phi\left(\frac{1}{t}\right), \quad 0 \leq t \leq w < \infty$$

$$\int_w^{\infty} e^{-y/2} y^{(6\alpha+6k-1)/12} |df(y)| < \infty$$

and

$$\int_w^{\infty} e^{-y/2} y^{(6\alpha-6k-13)/12} |f(y)| dy < \infty$$

Where $\phi(t)$ is a positive increasing function such that

$$\int_{c/n}^{\partial} \frac{\phi(t)}{t^2} dt = o\left\{ n \phi\left(\frac{1}{n}\right) \right\}, \quad n \rightarrow \infty$$

In Singh [6] estimated the order of function by harmonic means of the series (2) at the point $x = 0$. He proved the following:

Theorem: 3 For $\frac{5}{6} < \alpha < -\frac{1}{2}$

$$t_n(0) - f(0) = o\{\log n\}^{p+1}$$

Provided that

$$\int_t^{\partial} \frac{|\phi(y)|}{y^{\alpha+1}} dy = o\{\log 1/t\}^{1+p}, \quad t \rightarrow \infty, -1 < p < \infty$$

δ is fixed positive constant,

$$\int_{\partial}^n e^{y/2} y^{-(3\alpha+3)/4} |\phi(y)| dy = o\{n^{-(3\alpha+1)/4} (\log n)^{1+p}\}$$

And

$$\int_n^{\infty} e^{y/2} y^{-1/3} |\phi(y)| dy = o\{(\log n)^{p+1}\}, \quad n \rightarrow \infty$$

Jayaswal [3] generalized the result of Singh [6] and proved the following:

Theorem- 4 If for fixed ∂ and $\frac{5}{6} < \alpha < -\frac{1}{2}$

$$\phi(x) = \int_t^{\partial} \frac{|\phi(x)|}{x^{\alpha+1}} dx = o\left\{\log \frac{1}{t}\right\}^p, \quad t \rightarrow \infty$$

$$\int_{\partial}^n e^{\alpha/2} x^{-(2\alpha+3)/4} |\phi(x)| dx = o(\log n)^p$$

and

$$\int_n^{\infty} e^{\alpha/2} x^{-1/3} |\phi(x)| dx = o(\log n)^p$$

than

$$t_n(0) - f(0) = o(\log n)^p$$

where $\{p_n\}$ is a positive non increasing sequence such that

$$\sum_{v=0}^n \frac{p_{n-v}}{v+1} = o\left(\frac{p_n}{n}\right)$$

and t_n is the Nörlund mean

We prove our theorem for the Nörlund mean which is a more general than harmonic mean. In our theorem the range of α is increase to $-1 < \alpha < -\frac{1}{2}$, which is more useful for application .we prove the following theorem:

Main Theorem: If $\{p_n\}$ is a positive non increasing sequence of real number such that

$$\phi(t) = \int_0^t |\phi(y)| dy = o\{t^{\alpha+1} P(1/t)\}, \quad t \rightarrow \infty$$

$$\int_w^n e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\{n^{-(2\alpha-1)/4} P_n\}$$

and

$$\int_n^{\infty} e^{y/2} y^{-1/3} |\phi(y)| dy = o(P_n), \quad n \rightarrow \infty$$

Then for

$$-1 < \alpha < -\frac{1}{2}$$

$$t_n(0) - f(0) = o(P_n)$$

t_n is the Norlund mean of Laguerre expansion

Proof of the main theorem:

The relation $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$ we have

$$S_n(0) = \sum_{k=0}^n a_k L_k^{(\alpha)}(0) = \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) \sum_{k=0}^n L_k^{(\alpha)}(y) dy$$

$$= \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha+1)}(y) dy,$$

Therefore $t_n(0)$ is given by

$$(P_n)^{-1} \sum_{k=0}^n p_k \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_{n-k}^{(\alpha+1)}(y) dy.$$

Using orthogonal property of Laguerre polynomials and (15) we get

$$t_n(0) - f(0) = (P_n)^{-1} \sum_{k=0}^n p_k \int_0^\infty \phi(y) L_{n-k}^{(\alpha+1)}(y) dy$$

$$= \int_0^{c/n} + \int_{c/n}^w + \int_w^n + \int_n^\infty$$

$$= I_1 + I_2 + I_3 + I_4, \quad \text{Say}$$

Using orthogonal property and order estimates as given in Sezgo (1959), we get

$$I_1 = (P_n)^{-1} \sum_{k=0}^n p_k \cdot o(n-k)^{\alpha+1} \int_0^{c/n} |\phi(y)| dy$$

$$= (P_n)^{-1} \cdot P_n \cdot o(n^{\alpha+1}) \cdot o(n^{-\alpha-1} P_n)$$

$$= o(P_n), \quad \text{as } n \rightarrow \infty \tag{4}$$

Next,

$$I_2 = (P_n)^{-1} \sum_{k=0}^n p_k \cdot o(n-p)^{(2\alpha+1)/4} \int_{c/n}^w y^{-(2\alpha+3)/4} |\phi(y)| dy$$

Now

$$\sum_{k=0}^n p_k (n-k)^{(2\alpha+1)/4} = \left\{ \sum_{k=0}^{n/2} + \sum_{k=[n/2]+1}^n \right\} p_k (n-k)^{(2\alpha+1)/4}$$

$$= \{n - [n/2]\}^{(2\alpha+1)/4} P_{[n/2]} + \{p_{[n/2]}\} n^{(2\alpha+5)/4}$$

$$= o\{P_n \cdot n^{(2\alpha+1)/4}\}$$

Therefore

$$I_2 = o(n^{(2\alpha+1)/4}) \left[\int_{c/n}^w \{y^{-(2\alpha+3)/4} \Phi(y)\} dy + \int_{c/n}^w y^{-(2\alpha+7)/4} \Phi(y) dy \right]$$

$$\begin{aligned}
 &= o\left(n^{(2\alpha+1)/4}\right) \left[o(1) + o\left(n^{-(2\alpha+1)/4} P_n\right) + \int_{c/n}^w y^{-(2\alpha+3)/4} P\left(\frac{1}{y}\right) dy \right] \\
 &= o(1) + o(p_n) + o(p_n) n^{(2\alpha+1)/4} \int_{c/n}^w y^{(2\alpha-3)/4} dy \\
 &= o(1) + o(P_n) \\
 &= o(P_n)
 \end{aligned} \tag{5}$$

Next, I_3 can be written as

$$\begin{aligned}
 o(P_n)^{-1} \sum_{k=0}^n p_{n-k} \int_w^n e^{v/2} y^{-(2\alpha+3)/4} |\phi(y)| e^{v/2} y^{(2\alpha+3)/4} |L_n^{(\alpha+1)}(y)| dy \\
 &= o(P_n)^{-1} \sum_{k=0}^n p_{n-k} \cdot o\left(n^{(3\alpha+1)/4}\right) \int_w^n e^{v/2} y^{-(2\alpha+3)/4} |\phi(y)| dy \\
 &= o\left(n^{(3\alpha+1)/4}\right) \cdot o\left(n^{-(2\alpha+1)/4} P_n\right) \\
 &= o(P_n)
 \end{aligned} \tag{6}$$

Finally, considering I_4 we get

$$\begin{aligned}
 I_4 &= o(P_n)^{-1} \sum_{k=0}^n p_{n-k} \int_n^\infty e^{v/2} y^{-(3\alpha+5)/4} |\phi(y)| \times e^{-v/2} y^{(3\alpha+5)/6} |L_n^{(\alpha+1)}(y)| dy \\
 &= o(P_n)^{-1} \sum_{k=0}^n p_{n-k} \cdot o\left(k^{(\alpha+1)/2}\right) \int_n^\infty \frac{e^{v/2} y^{-1/3} |\phi(y)|}{y^{(\alpha+1)/2}} dy \\
 &= o(P_n)^{-1} \cdot o(P_n) \cdot o\left(n^{(\alpha+1)/2}\right) \cdot n^{-(\alpha+1)/2} \cdot o(P_n)
 \end{aligned}$$

Therefore,

$$I_4 = o(P_n) \tag{7}$$

Therefore from (4), (5),(6),(7) we have

$$t_n(0) - f(0) = o(P_n)$$

Which complete the proof of our theorem.

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