Research Journal of Pure Algebra -1(2), May - 2011, Page: 35-39

RJPA Available online through www.rjpa.info ISSN 2248 – 9037

DEGREE OF APPROXIMATION BY NÖRLUND SUMMABILITY MEANS OF LAGUERRE SERIES

Sandeep K. Tiwari[1] and Dinesh K. Kachhara*[2]

[1]School of Studies in Mathematics, Vikram University, Ujjain

[2]Mandsaur Institute of Pharmacy, Mandsaur (M.P.) 458001

***E-mail:** dkkachhara@rediffmail.com*

(Received on: 05-02-11; Accepted on: 13-03-11)

ABSTRACT

In this paper a general result on degree of Approximation of the Nörlund summability means of Fourier series with a weight has been obtained at the point x = 0 by Nörlund mean.

In 1981, S.N. Lal and K. N.Singh [4] obtained on the Absolute Summability of Fourier Series by Nörlund mean.

In this paper, we obtain the comparable result of [4] with degree of approximation by Nörlund summability of weighted Fourier series at the point x=0 by Nörlund mean.

***Key word:** Nörlund mean, Cesàro means, Harmonic means, Approximation of a Function.*

INTRODUCTION:

Let $\sum u_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants and let us write

$$P(n) = P_n, \quad \text{where} \quad P_n = \sum_{k=0}^{n} p_k .$$

The sequence-to sequence transformation

$$t_n = \sum_{k=0}^{n} \frac{p_{n-k} S_k}{p_n} = \sum \frac{p_k S_{n-k}}{p_n}, \quad p_n \neq 0 \tag{1}$$

and the sequence $\{t_n\}$ of Nörlund mean of the sequence $\{S_n\}$ generated by the sequence of coefficient $\{p_n\}$.

The important particular cases of the Nörund means are:

1. Harmonic means when $p_k = \dfrac{1}{k+1}$
2. Cesàro mean, when $p_k = \begin{pmatrix} k+\delta-1 \\ \delta-1 \end{pmatrix}, \quad \delta > 0.$
3. Nörlund mean (N, p_n) when $q_n = 1$ for all n.
4. Nörlund mean (N, q_n) when $p_n = 1$ for all n.

The Laguerre expansion of a function $f(x) \in L(0,\infty)$ is given by

---

*Corresponding author: Dinesh Kumar Kachhara, *E-mail: dkkachhara@rediffmail.com

$$f(x)=\sum_{n=0}^{\infty}a_n L_n^{(\alpha)}(x) \tag{2}$$

Where

$$a_n=\left\{\Gamma(\alpha+1)\binom{n+\alpha}{n}\right\}^{-1}\int_0^{\infty}e^{-v}y^{\alpha}f(y)L_n^{(\alpha)}(y)dy \tag{3}$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre expansion of order $\alpha>-1$, defined by the function

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)w^n=(1-w)^{-n-1}\exp\left(\frac{-xw}{1-w}\right)$$

And existence of the integral (3) is presumed.

We write

$$\phi(y)=\{\Gamma(\alpha+1)\}^{-1}e^{-v}y^{\alpha}\{f(y)-f(0)\}$$

Gupta [2] estimated the order of the function by Cesàro means of the series (2) at the point $x=0$ after replacing the continuity condition in Szego's theorem [7] by a much lighter condition.

Theorem: 1 If

$$F(t)=\int_0^t\frac{|f(y)|}{y}dy=o\left\{\log\frac{1}{t}\right\}^{1+p},t\to 0,-1<p<\infty$$

and

$$\int_1^{\infty}e^{-y/2}y^{(3\alpha-3k-1)/3}|f(y)|dy<\infty$$

Then

$$\sigma_n^k(0)=o(\log n)^{p+1}$$

Provided $k>\alpha+\dfrac{1}{2},\alpha>-1,\sigma_n^k(0)$ being the nth Cesàro mean of order k

Theorem: 2 For $k>\alpha>-1$

$$\sigma_n^k(f,0)=o(n^{-1/4})+o\{\phi(1/n)\}$$

Provided

$$\int_0^t|df(y)|\le A\phi(1/t), \qquad 0\le t\le w<\infty$$

$$\int_w^{\infty}e^{-v/2}y^{(6\alpha+6k-1)/12}|df(y)|<\infty$$

and

$$\int_w^{\infty}e^{-v/2}y^{(6\alpha-6k-13)/12}|f(y)|dy<\infty$$

Where $\phi(t)$ is a positive increasing function such that

$$\int_{c/n}^{\partial}\frac{\phi(t)}{t^2}dt=o\{n\phi(1/n)\}, \qquad n\to\infty$$

In Singh [6] estimated the order of function by harmonic means of the series (2) at the point $x = 0$. He proved the following:

Theorem: 3 For $\frac{5}{6} < \alpha < -\frac{1}{2}$

$$t_n(0) - f(0) = o\{\log n\}^{p+1}$$

Provided that

$$\int_t^{\partial} \frac{|\phi(y)|}{y^{\alpha+1}} dy = o\{\log 1/t\}^{1+p}, \qquad t \to \infty, -1 < p < \infty$$

δ is fixed positive constant,

$$\int_{\partial}^{n} e^{v/2} y^{-(3\alpha+3)/4} |\phi(y)| dy = o\{n^{-(3\alpha+1)/4} (\log n)^{1+p}\}$$

And

$$\int_n^{\infty} e^{v/2} y^{-1/3} |\phi(y)| dy = o\{(\log n)^{p+1}\}, \qquad n \to \infty$$

Jayaswal [3] generalized the result of Singh [6] and proved the following:

Theorem- 4 If for fixed ∂ and $\frac{5}{6} < \alpha < -\frac{1}{2}$

$$\phi(x) = \int_t^{\partial} \frac{|\phi(x)|}{x^{\alpha+1}} dx = o\left\{\log \frac{1}{t}\right\}^p, \qquad t \to \infty$$

$$\int_{\partial}^{n} e^{\alpha/2} x^{-(2\alpha+3)/4} |\phi(x)| dx = o(\log n)^p$$

and

$$\int_n^{\infty} e^{\alpha/2} x^{-1/3} |\phi(x)| dx = o(\log n)^p$$

than

$$t_n(0) - f(0) = o(\log n)^p$$

where $\{p_n\}$ is a positive non increasing sequence such that

$$\sum_{v=0}^{n} \frac{p_{n-v}}{v+1} = o\left(\frac{p_n}{n}\right)$$

and t_n is the Nörlund mean

We prove our theorem for the Nörlund mean which is a more general than harmonic mean. In our theorem the range of α is increase to $-1 < \alpha < -\frac{1}{2}$, which is more useful for application .we prove the following theorem:

Main Theorem: If $\{p_n\}$ is a positive non increasing sequence of real number such that

$$\phi(t) = \int_0^t |\phi(y)| dy = o\{t^{\alpha+1} P(1/t)\}, \qquad t \to \infty$$

$$\int_w^n e^{v/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\{n^{-(2\alpha-1)/4} P_n\}$$

and

$$\int_n^{\infty} e^{v/2} y^{-1/3} |\phi(y)| dy = o(P_n), \qquad n \to \infty$$

Then for

$$-1<\alpha<-\frac{1}{2}$$
$$t_n(0)-f(0)=o(P_n)$$

t_n is the Norlund mean of Laguerre expansion

Proof of the main theorem:

The relation $L_n^{(\alpha)}(0)=\binom{n+\alpha}{n}$ we have

$$S_n(0)=\sum_{k=0}^{n}a_k L_k^{(\alpha)}(0)=\{\Gamma(\alpha+1)\}^{-1}\int_0^\infty e^{-v}y^\alpha f(y)\sum_{k=0}^{n}L_\alpha^{(\alpha)}(y)dy$$
$$=\{\Gamma(\alpha+1)\}^{-1}\int_0^\infty e^{-v}y^\alpha f(y)L_n^{(\alpha+1)}(y)dy,$$

Therefore $t_n(0)$ is given by

$$(P_n)^{-1}\sum_{k=0}^{n}p_k\{\Gamma(\alpha+1)\}^{-1}\int_0^\infty e^{-v}y^\alpha f(y)L_{n-k}^{(\alpha+1)}(y)dy.$$

Using orthogonal property of Laguerre polynomials and (15) we get

$$t_n(0)-f(0)=(P_n)^{-1}\sum_{k=0}^{n}p_k\int_0^\infty \phi(y)L_{n-k}^{(\alpha+1)}(y)dy$$
$$=\int_0^{c/n}+\int_{c/n}^{w}+\int_w^n+\int_n^\infty$$
$$=I_1+I_2+I_3+I_4, \qquad \text{Say}$$

Using orthogonal property and order estimates as given in Sezgo (1959), we get

$$I_1=(P_n)^{-1}\sum_{k=0}^{n}p_k.o(n-k)^{\alpha+1}\int_0^{c/n}|\phi(y)|dy$$
$$=(P_n)^{-1}.P_n.o(n^{\alpha+1}).o(n^{-\alpha-1}P_n)$$
$$=o(P_n), \quad \text{as} \quad n\to\infty \tag{4}$$

Next,

$$I_2=(P_n)^{-1}\sum_{k=0}^{n}p_k.o(n-p)^{(2\alpha+1)/4}\int_{c/n}^{w}y^{-(2\alpha+3)/4}|\phi(y)|dy$$

Now

$$\sum_{k=0}^{n}p_k(n-k)^{(2\alpha+1)/4}=\left\{\sum_{k=0}^{n/2}+\sum_{[n/2]+1}^{n}\right\}p_k(n-k)^{(2\alpha+1)/4}$$
$$=\{n-[n/2]\}^{(2\alpha+1)/4}P_{[n/2]}+\{p_{[n/2]}\}n^{(2\alpha+5)/4}$$
$$=o\{P_n.n^{(2\alpha+1)/4}\}.$$

Therefore

$$I_2=o(n^{(2\alpha+1)/4})\left[\{y^{-(2\alpha+3)/4}\Phi(y)\}_{c/n}^{w}+\int_{c/n}^{w}y^{-(2\alpha+7)/4}\Phi(y)dy\right]$$

$$= o\left(n^{(2\alpha+1)/4}\right)\left[o(1)+o\left(n^{-(2\alpha+1)/4}P_n\right)+\int_{c/n}^{w} y^{-(2\alpha+3)/4}P\left(1/y\right)dy\right]$$

$$= o(1)+o(p_n)+o(p_n).n^{(2\alpha+1)/4}\int_{c/n}^{w} y^{(2\alpha-3)/4}dy$$

$$= o(1)+o(P_n)$$

$$= o(P_n) \tag{5}$$

Next, I_3 can be written as

$$o(P_n)^{-1}\sum_{k=0}^{n} p_{n-k}\int_{w}^{n} e^{v/2}y^{-(2\alpha+3)/4}|\phi(y)|e^{v/2}y^{(2\alpha+3)/4}\left|L_n^{(\alpha+1)}(y)\right|dy$$

$$= o(P_n)^{-1}\sum_{k=0}^{n} p_{n-k}.o\left(n^{(3\alpha+1)/4}\right)\int_{w}^{n} e^{v/2}y^{-(2\alpha+3)/4}|\phi(y)|dy$$

$$= o\left(n^{(3\alpha+1)/4}\right).o\left(n^{-(2\alpha+1)/4}P_n\right)$$

$$= o(P_n) \tag{6}$$

Finally, considering I_4 we get

$$I_4 = o(P_n)^{-1}\sum_{k=0}^{n} p_{n-k}\int_{n}^{\infty} e^{v/2}y^{-(3\alpha+5)/4}|\phi(y)| \times e^{-v/2}y^{(3\alpha+5)/6}\left|L_n^{(\alpha+1)}(y)\right|dy$$

$$= o(P_n)^{-1}\sum_{k=0}^{n} p_{n-k}.o\left(k^{(\alpha+1)/2}\right)\int_{n}^{\infty}\frac{e^{v/2}y^{-1/3}|\phi(y)|}{y^{(\alpha+1)/2}}dy$$

$$= o(P_n)^{-1}.o(P_n).o\left(n^{(\alpha+1)/2}.n^{-(\alpha+1)/2}\right).o(P_n)$$

Therefore,

$$I_4 = o(P_n) \tag{7}$$

Therefore from (4), (5),(6),(7) we have

$$t_n(0)-f(0)=o(P_n)$$

Which complete the proof of our theorem.

REFERENCES:

[1] Beohar, B.K. and Jadiya, B.L.: Degree of approximation by Cesàro mean of Fourier Laguerre series. *J. pure apppl. Math*. 11, 1162-65, 1980.

[2] Gupta,D.P.: Degree of approximation by Cesàro mean of Fourier –Laguerre expansion, *Acta Sci.Math.*,32,255-259,1971.

[3] Jayaswal, A.K.: Approximation of a Function by Nörlund mean of Fourier-Laguerre expansion. *Vijnon Shodh Bharti*, 1, 33-41, 1978.

[4] Lal, S.N.and Singh, K. N.: On the absolute summability of weighted Fourier series by Nörlund means,Indian *J. Pure Math*.,12(3)350-361,1981.

[5] Lal, Shyam and Nigam H.K.: Degree of Approximation by *(N,p,q)* summability means of the Fourier – Laguerre expansion Tamkang J. of Mathematics,32,143-149,2001.

[6] Singh, T.: Degree of approximation by harmonic means of Fourier-Laguerre expansion.*Publ.Math.Debrecen*, 24, 53-57, 1977.

[7] Szegö, G., Orthogonal Polynomials, Colloq.Publ.Amer.Math.Soc.,New York,1959.