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DUAL HEYTING ALMOST DISTRIBUTIVE L

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ABSTRACT

In this paper, the concept of a dual Heyting Almost Distributive Lattice (Dual H-ADL) as a generalization of a dual Heyting algebra in the class of ADLs is introduced and studied its properties. We characterize a Dual H-ADL in terms of its principal ideals. Necessary and sufficient conditions are derived. It is shown that every dual H-ADL is Dual Pseudo-Complemented Almost Distributive Lattice.

AMS Mathematical subject classification (2000): *06D20, 06D99.*

Key Words: *Almost Distributive Lattice (ADL); Dual Heyting Algebra; Maximal element; Principal ideal; Dual Heyting Almost Distributive Lattice (Dual H-ADL).*

1. INTRODUCTION

Heyting algebra is a relatively pseudo-complemented distributive lattice which arises from non-classical logic and was first investigated by T. Skolem about 1920. It was named as Heyting algebra after the Dutch Mathematician Arend Heyting. Heyting algebras are less often called pseudo-Boolean algebras. It was introduced by Birkhoff G. under a different name Brouwerian lattice and with a different notation. Birkhoff further developed the theory of Heyting algebras from a lattice theoretic point of view. Since then Heyting algebras, also called pseudo-complemented distributive lattices with 0, have been studied quite extensively. Later H. B. Curry about 1963 developed the theory of Heyting algebras.

Heyting algebras generalize Boolean algebras in the sense that Heyting algebra satisfying $a \to a = 1$ is a Boolean algebra. Heyting algebras serve as the algebraic models of propositional intutionstic logic in the same way Boolean algebras model propositional classical logic. Complete Heyting algebras are a central object of study in pointless topology. The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and G. C. Rao as a common abstraction to most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the distributive lattices on the other. For this reason, G.C. Rao, Berhanu and Ratna Mani [4] introduced the concept of Heyting Almost Distributive Lattices (H-ADL) as generalization of Heyting algebra and derived many important results. Unlike in lattices, the dual of an ADL in not an ADL in general. For this reason, in this paper, we introduce the concept of a Dual Heyting Almost Distributive Lattice (dual H-ADL) and derive a number of important laws and results satisfied by a dual H-ADL. We also characterize a dual H-ADL in terms of the lattice of all of its principal ideals.

2. PRELIMINARIES

In this section, we give the necessary definitions and important properties of an ADL taken from [7] for ready reference.

Definition 2.1 [7] An algebra $(A, \vee, \wedge, 0$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms:

i. $x \vee 0 = x$
ii. $0 \wedge x = 0$
iii. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
iv. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
v. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
vi. $(x \vee y) \wedge y = y$ for all $x, y, z \in A$.

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A non-empty subset I of an ADL A is called an ideal of A if $x \vee y \in I$ and $x \wedge a \in I$ for any $x, y \in I$ and $a \in A.$ The principal ideal of A generated by x is denoted by $(x].$ The set $PI(A)$ of all principal ideals of A forms a distributive lattice under the operations $\vee, \wedge$ defined by $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$ in which $(0]$ is the least element. If A has a maximal element $m,$ then $(m]$ is the greatest element of $PI(A)$.

Theorem 2.2. [7] Let A be an ADL and $x, y \in A$. Then the following are equivalent:
(i) $\left(x\right] \subseteq \left(y\right]$
(ii) $y \wedge x = x$
(iii) $y \vee x = y$
(iv) $\left[y\right) \subseteq \left[x\right).$
For other properties of an ADL, we refer to [7].

If A is a bounded distributive lattice, then the set of all complemented elements of A forms a Boolean algebra and it is called the (Birkhoff) center of $A.$ This concept is extended to an ADL in [6] as follows.

Definition 2.3. [6] Let A be an ADL with a maximal element m and $B(A) = \{a \in A \mid a \wedge b = 0$ and $a \vee b$ is a maximal for some $b \in A\}.$ Then $(B(A), \vee, \wedge)$ is a relatively complemented ADL and it is called the Birkhoff center of $A.$ We use the symbol B instead of $B(A)$ when there is no ambiguity.

For any $b \in B, b \wedge m$ is a complemented element in the distributive lattice $[0, m]$ whose complement will be denoted by $(b \wedge m)'.$

3. DUAL $H-$ADL

We begin with the following definition of a Heyting algebra taken from [2].

Definition 3.1. [2] A bounded distributive lattice $(A, \vee, \wedge, 0, 1)$ is said to be Heyting algebra if there exists a binary operation $\rightarrow$ on A such that, for any $x, y, z \in A, x \wedge z \leq y$ if and only if $z \leq x \rightarrow y.$

Since the dual A^d of a distributive lattice $(A, \vee, \wedge)$ is again a distributive lattice. We give the following definition.

Definition 3.2. A distributive lattice $(A, \vee, \wedge)$ is called a dual Heyting algebra if its dual A^d of A is a Heyting algebra.

Theorem 3.3. A distributive lattice $(A, \vee, \wedge, 0, 1)$ is a dual Heyting algebra if and only if there exists a binary operation $\leftarrow$ satisfying the following:
(i) $x \leftarrow x = 0$
(ii) $(x \leftarrow y) \vee y = y$
(iii) $x \vee (x \leftarrow y) = x \vee y$
(iv) $z \leftarrow (x \vee y) = (z \leftarrow x) \vee (z \leftarrow y)$
(v) $(x \wedge y) \leftarrow z = (x \leftarrow z) \vee (y \leftarrow z)$

In [4], G.C. Rao and Berhanu extended the concept of Heyting algebra to the class of ADLs as follows.

Definition 3.4. Let $(A, \vee, \wedge, 0, m)$ be an ADL with a maximal element $m.$ Suppose $\rightarrow$ is a binary operation on A satisfying the following conditions: for all $x, y, z \in A,$
(i) $x \rightarrow x = m$
(ii) $(x \rightarrow y) \wedge y = y$
(iii) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$
(iv) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

(v) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z).$

Then $(A, \vee, \wedge, \rightarrow, 0, m)$ is called a Heyting Almost Distributive Lattice (H-ADL).

It was observed in [4], that an ADL $(A, \vee, \wedge, 0, m)$ is a H-ADL with the binary operation $\rightarrow$ if and only if to each $a \in A,$ the distributive lattice $([0,a], \vee, \wedge, 0, a)$ is a Heyting algebra with the binary operation $\rightarrow_a$ defined by $x \rightarrow_a y = (x \rightarrow y) \wedge a$ for any $x, y \in [0,a].$

In general, the dual of an ADL is not an ADL. For this reason, in this paper, we introduce the concept of dual Heyting Almost Distributive Lattice (dual H-ADL) as follows.

Definition 3.5. An ADL $(A, \vee, \wedge, 0$ with a maximal element m is called a dual Heyting Almost Distributive Lattice (dual H-ADL), if to each $a \in A,$ the distributive lattice $([o,a], \vee, \wedge, 0, m)$ is a dual Heyting algebra with respect to the binary operation denoted by $\leftarrow_a$.

Theorem 3.6. Let A be an ADL with a maximal element $m.$ Then A is a dual H-ADL if and only if to each $a \in A,$ there exists a binary operation $\leftarrow^a$ on A satisfying the following conditions:

(i) $x \leftarrow^a x = 0$

(ii) $[(x \leftarrow^a y) \vee y] \wedge a = y \wedge a$

(iii) $[x \vee (x \leftarrow^a y)] \wedge a = (x \vee y) \wedge a$

(iv) $z \leftarrow^a (x \vee y) = (z \leftarrow^a x) \vee (z \leftarrow^a y)$

(v) $(x \wedge y) \leftarrow^a z = (x \leftarrow^a z) \vee (y \leftarrow^a z)$ for all $x, y, z \in A.$

Proof. Let A be a dual H-ADL and $a \in A.$ Then $([0,a], \vee, \wedge, \leftarrow_a, 0, a)$ is a dual Heyting algebra. Define a binary operation $\leftarrow^a$ on A by $x \leftarrow^a y = (x \wedge a) \leftarrow_a (y \wedge a)$ for all $x, y \in A.$ Let $x, y \in A.$ Then

(a) $x \leftarrow^a x = (x \wedge a) \leftarrow_a (x \wedge a) = 0$

(b) $[(x \leftarrow^a y) \vee y] \wedge a = [(x \wedge a) \leftarrow_a (y \wedge a)] \vee (y \wedge a) = y \wedge a$

(c) $[x \vee (x \leftarrow^a y)] \wedge a = (x \wedge a) \vee [(x \wedge a) \leftarrow_a (y \wedge a)] = (x \wedge a) \vee (y \wedge a) = (x \vee y) \wedge a$

(d) $z \leftarrow^a (x \vee y) = (z \wedge a) \leftarrow_a [(x \vee y) \wedge a] = (z \wedge a) \leftarrow_a (x \wedge a) \vee (z \wedge a) \leftarrow_a (y \wedge a)$

$$= (z \leftarrow^a x) \vee (z \leftarrow^a y)$$

(e) $(x \wedge y) \leftarrow^a z = (x \wedge y \wedge a) \leftarrow_a (z \wedge a) = [(x \wedge a) \leftarrow_a (z \wedge a)] \vee [(y \wedge a) \leftarrow_a (z \wedge a)]$

$$= (x \leftarrow_a z) \vee (y \leftarrow_a z)$$

Conversely, suppose to each $a \in A,$ there exists a binary operation $\leftarrow^a$ satisfying conditions (a) to (e). For $x, y \in [0,a],$ define $x \leftarrow_a y = (x \leftarrow^a y) \wedge a.$ Let $x, y, z \in [0,a].$ Then

(i) $x \leftarrow_a x = (x \leftarrow^a x) \wedge a = 0.$

(ii) $(x \leftarrow_a y) \vee y = ((x \leftarrow^a y) \wedge a) \vee (y \wedge a) = [(x \leftarrow^a y) \vee y] \wedge a = y \wedge a = y.$

(iii) $x \vee (x \leftarrow_a y) = [(x \wedge a) \vee (x \leftarrow^a y) \wedge a)] = [x \vee (x \leftarrow^a y)] \wedge a = x \vee y.$

(iv) $z \leftarrow_a \{(x \vee y)\} = [z \leftarrow^a (x \vee y)] \wedge a = \{[z \leftarrow^a x] \vee [z \leftarrow^a y]\} \wedge a$

$$= \{[z \leftarrow^a x] \wedge a\} \vee \{[z \leftarrow^a y] \wedge a\}$$

$$= (z \leftarrow^a x) \vee (z \leftarrow^a y).$$

(v) $(x \wedge y) \leftarrow_a z = [(x \wedge y) \leftarrow^a z] \wedge a = [(x \leftarrow^a z) \vee (y \leftarrow^a z)] \wedge a$

$$= (x \leftarrow_a z) \vee (y \leftarrow_a z).$$

Therefore, by Theorem 3.3, $[0,a]$ is a dual Heyting algebra for all $a \in A$ and hence A is a dual H-ADL.

Theorem 3.7. Let A be an ADL with a maximal element $m.$ Then the following are equivalent:
(i) A is a dual H-ADL.
(ii) $[0, m]$ is a dual Heyting algebra.
(iii) There exists a binary operation $\leftarrow$ on A satisfying the following conditions:

d_1 $x \leftarrow x = 0$
d_2 $[(x \leftarrow y) \vee y] \wedge m = y \wedge m$
d_3 $[x \vee (x \leftarrow y)] \wedge m = (x \vee y) \wedge m$
d_4 $z \leftarrow (x \vee y) = (z \leftarrow x) \vee (z \leftarrow y)$
d_5 $(x \wedge y) \leftarrow z = (x \leftarrow z) \vee (y \leftarrow z)$ for all $x, y, z \in A.$

Proof. (i)$\Rightarrow$(ii) follows from Definition 3.5 and (ii)$\Rightarrow$(iii) follows from Theorem 3.6 by taking $x \leftarrow y = (x \wedge m) \leftarrow^m (y \wedge m)$ for all $x, y \in A.$ Now assume (iii). For any $a \in A,$ we know that $[0, a]$ is a distributive lattice. For $x, y \in [0, a],$ define $x \leftarrow_a y = (x \leftarrow y) \wedge a.$ Then it can be verified that $([0, a], \vee, \wedge, \leftarrow_a, 0, a)$ is a dual Heyting algebra and hence A is a dual H-ADL.

Note 3.8. Let A be a dual H-ADL. Then, by Theorem 3.7, $([0, m], \vee, \wedge, \leftarrow_m, 0, m)$ is a dual Heyting algebra. From now onwards, for $x, y \in A,$ define $(x \wedge m) \leftarrow_m (y \wedge m)$ by $x \leftarrow y.$ So that for any $x, y, z \in A,$ $(x \vee z) \wedge m \geq y \wedge m$ if and only if $z \wedge m \geq (x \leftarrow y) \wedge m.$

Here afterwards, A stands for a dual H-ADL with a maximal element $m.$ In the following theorem, we derive some preliminary properties of a dual H-ADL.

Theorem 3.9. For any $x, y, z \in A,$ we have the following:
(i) $(x \wedge y) \leftarrow y = x \leftarrow y$
(ii) $m \leftarrow x = 0$
(iii) $(0 \leftarrow x) \wedge m = x \wedge m$
(iv) $x \leftarrow (x \vee y) = x \leftarrow y$
(v) $(x \wedge y) \leftarrow (x \vee y) = (y \leftarrow x) \vee (x \leftarrow y).$

Proof. Let $x, y \in A.$ Then $(x \wedge y) \leftarrow y = (x \leftarrow y) \vee (y \leftarrow y) = x \leftarrow y.$ Thus we get (i). To prove (ii), $0 = x \leftarrow x = (m \wedge x) \leftarrow x \quad = m \leftarrow x.$ Since $x \leftarrow x = 0,$ we get $x \leftarrow (x \vee y) = x \leftarrow y.$ To prove (iv), $x \leftarrow (x \vee y) = (x \leftarrow x) \vee \ (x \leftarrow y) = x \leftarrow y.$ Finally (v) follows from (iii) and (iv).

Theorem 3.10. Let $(A, \vee, \wedge, \leftarrow, 0, m)$ be a dual H-ADL with a maximal element $m.$ Then, for any maximal element n in A, $(A, \vee, \wedge, \leftarrow_n, 0, n)$ is a dual H-ADL where for any $x, y \in A, x \leftarrow_n y = (x \leftarrow y) \wedge n.$

Proof. Let $(A, \vee, \wedge, \leftarrow, 0, m)$ be a dual H-ADL and n be a maximal element in $A.$ Since $x \vee (x \leftarrow_n y) =$ $[x \vee (x \leftarrow y)] \wedge m \wedge n \geq y \wedge m \wedge n = y \wedge n,$ we get $x \vee (x \leftarrow_n y) \geq y \wedge n.$ Suppose $c \in A$ such that $(x \vee c) \wedge n \geq y \wedge n.$ Then $(x \vee c) \wedge n \wedge m \geq y \wedge n \wedge m.$ Thus $(x \vee c) \wedge m \geq y \wedge m.$ By definition , $c \wedge m \geq$ $(x \leftarrow y) \wedge m$ and hence $c \wedge n \geq (x \leftarrow y) \wedge n = (x \leftarrow_n y).$ Therefore $(A, \vee, \wedge, \leftarrow_n, 0, n)$ is a dual H-ADL. In the following theorem, we prove some important properties of dual H-ADL.

Theorem 3.11 For any $x, y, z \in A,$ we have the following:
(i) If $x \leq y,$ then $z \leftarrow x \leq z \leftarrow y$ and $x \leftarrow z \geq y \leftarrow z.$
(ii) If $y \leq x,$ then $x \leftarrow y = 0.$
(iii) $(x \vee y) \leftarrow y = (x \vee y) \leftarrow x = 0$
(iv) $(x \vee y) \wedge m = (x \vee z) \wedge m \Leftrightarrow x \leftarrow y = x \leftarrow z$.

Proof. Let $x, y, z \in A$. Suppose $x \le y$. Then $z \leftarrow (x \vee y) = z \leftarrow y = (z \leftarrow x) \vee (z \leftarrow y)$ and hence $z \leftarrow x \le z \leftarrow y$. Again, since $x \le y$, we get $(y \wedge x) \leftarrow z = x \leftarrow z = (y \leftarrow z) \vee (x \leftarrow z)$ and hence $x \leftarrow z \ge y \leftarrow z$. Thus we get (i). Suppose $y \le x$. Then by (i) above, we get $0 = y \leftarrow y \ge x \leftarrow y$. To prove (iii), since $x \wedge m \le (x \vee y) \wedge m$ and $y \wedge m \le (x \vee y) \wedge m$, we get $(x \vee y) \leftarrow x = 0$ and $(x \vee y) \leftarrow y = 0$.

Finally, suppose $(x \vee y) \wedge m = (x \vee z) \wedge m$. Then $x \leftarrow [(x \vee y) \wedge m] = x \leftarrow [(x \vee z) \wedge m]$ and hence $x \leftarrow y = x \leftarrow z$. Conversely, suppose $x \leftarrow y = x \leftarrow z$. Then $[x \vee (x \leftarrow y)] \wedge m = [x \vee (x \leftarrow z)] \wedge m$ and hence $(x \vee y) \wedge m = (x \vee z) \wedge m$. Thus we get (iv).

Theorem 3.12. For any $x, y, z \in A$, we have the following:

(i) $(x \wedge m) \leftarrow y = x \leftarrow y$.

(ii) $(x \leftarrow y) \wedge m = (x \leftarrow (y \wedge m)) \wedge m$.

(iii) $((x \wedge m) \leftarrow (y \wedge m)) \wedge m = (x \leftarrow y) \wedge m$.

(iv) $(z \leftarrow (x \vee y)) \wedge m = (z \leftarrow (y \vee x)) \wedge m$.

(v) $(z \leftarrow (x \wedge y)) \wedge m = (z \leftarrow (y \wedge x)) \wedge m$.

Proof. Let $x, y, z \in A$. Then $(x \wedge m) \leftarrow y = (x \leftarrow y) \vee (m \leftarrow y) = (x \leftarrow y) \vee 0 = x \leftarrow y$. Thus we get (i). Since $[x \vee (x \leftarrow y)] \wedge m \ge y \wedge m$, we get $(x \leftarrow y) \wedge m \ge (x \leftarrow (y \wedge m)) \wedge m$. On the other hand, since $\{x \vee [x \leftarrow (y \wedge m)]\} \wedge m \ge y \wedge m$, we get $(x \leftarrow (y \wedge m)) \wedge m \ge (x \leftarrow y) \wedge m$. Thus we get (ii). (iii) follow from (i) and (ii). Since $(z \leftarrow (x \vee y)) \wedge m = \{z \leftarrow [(x \vee y) \wedge m]\} \wedge m = \{z \leftarrow [(y \vee x) \wedge m]\} \wedge m = (z \leftarrow (y \vee x)) \wedge m$.

Thus we get (iv). Finally, since $x \wedge y \wedge m = y \wedge x \wedge m$, we get $(z \leftarrow (x \wedge y)) \wedge m = (z \leftarrow (y \wedge x)) \wedge m$ and hence the theorem.

Theorem 3.13. For any $x, y, z \in A$, we have the following:

(i) $y \wedge m \ge [x \leftarrow (x \leftarrow y)] \wedge m$

(ii) $x \wedge m \ge (y \leftarrow z) \wedge m \Leftrightarrow y \wedge m \ge (x \leftarrow z) \wedge m$

(iii) $[x \leftarrow (y \leftarrow z)] \wedge m = [(x \vee y) \leftarrow z] \wedge m$

(iv) $[x \leftarrow (y \leftarrow z)] \wedge m = [y \leftarrow (x \leftarrow z)] \wedge m$

(v) $x \leftarrow (y \leftarrow x) = 0$.

Proof. Since $[(x \leftarrow y) \vee y] \wedge m \ge y \wedge m$, we get $y \wedge m \ge [(x \leftarrow y) \leftarrow y] \wedge m$ and hence we get (i). Suppose $x \wedge m \ge (y \leftarrow z) \wedge m$. Then $(y \vee x) \wedge m \ge z \wedge m$ and hence $y \wedge m \ge (x \leftarrow z) \wedge m$. By symmetry, we get the converse. To prove (iii), since $[(x \vee y) \vee \{x \leftarrow (y \leftarrow z)\}] \wedge m = [y \vee \{x \vee (x \leftarrow (y \leftarrow z))\}] \wedge m =$ $[y \vee \{x \vee (y \leftarrow z)\}] \wedge m = [x \vee \{y \vee (y \leftarrow z)\}] \wedge m = [x \vee y \vee z] \wedge m \ge z \wedge m$, we get $(x \leftarrow (y \leftarrow z)) \wedge m \ge [(x \vee y) \leftarrow z] \wedge m$. On the other hand, since $((x \vee y) \vee ((x \vee y) \leftarrow z)) \wedge m \ge z \wedge m$, we get $(x \vee ((x \vee y) \leftarrow z)) \wedge m \ge (y \leftarrow z) \wedge m$ and hence $((x \vee y) \leftarrow z) \wedge m \ge (x \leftarrow (y \leftarrow z)) \wedge m$.

Thus we get result. Since $((x \vee y) \leftarrow z) \wedge m = ((y \vee x) \leftarrow z) \wedge m$, we get $(x \leftarrow (y \leftarrow z)) \wedge m = (y \leftarrow (x \leftarrow z)) \wedge m$.

Thus we get (iv). Finally, $(x \leftarrow (y \leftarrow x)) \wedge m = ((x \vee y) \leftarrow x) \wedge m = 0 \wedge m = 0$ (since by Theorem 3.11.(iii)).

In general, the operation $\leftarrow$ is not compatible with either $\vee$ or $\wedge$ in A. In the following theorem, we derive partial compatibility.

Theorem 3.14. For any $x, y, z, w \in A$, we have the following:

(i) $((x \leftarrow y) \vee (z \leftarrow w)) \wedge m \ge ((x \vee z) \leftarrow (y \vee w)) \wedge m$

(ii) $((x \leftarrow y) \vee (z \leftarrow w)) \wedge m \ge ((x \wedge z) \leftarrow (y \wedge w)) \wedge m$

(iii) $(((x \leftarrow y) \leftarrow y) \leftarrow y) \wedge m = (x \leftarrow y) \wedge m$
(iv) $(x \leftarrow y) \wedge m \geq ((z \leftarrow x) \leftarrow (z \leftarrow y)) \wedge m$
(v) $(x \leftarrow y) \wedge m \geq ((y \leftarrow z) \leftarrow (x \leftarrow z)) \wedge m$
(vi) $((x \leftarrow z) \wedge (y \leftarrow z)) \wedge m \geq ((x \vee y) \leftarrow z) \wedge m$
(vii) $((z \leftarrow x) \wedge (z \leftarrow y)) \wedge m \geq (z \leftarrow (x \wedge y)) \wedge m.$

Proof. Since $((x \vee z) \vee (x \leftarrow y) \vee (z \leftarrow w)) \wedge m \geq (y \vee w) \wedge m,$ we get $((x \leftarrow y) \vee (z \leftarrow w)) \wedge m \geq ((x \vee z) \leftarrow (y \vee w)) \wedge m.$ Thus we get (i).

Since $((x \wedge z) \vee ((x \leftarrow y) \vee (z \leftarrow w))) \wedge m = (x \vee (x \leftarrow y) \vee (z \leftarrow w)) \wedge (z \vee (x \leftarrow y) \vee (z \leftarrow w)) \geq (y \wedge w) \wedge m,$ we get $((x \leftarrow y) \vee (z \leftarrow w)) \wedge m \geq ((x \wedge z) \leftarrow (y \wedge w)) \wedge m.$ Since $((x \leftarrow y) \vee ((x \leftarrow y) \leftarrow y)) \wedge m \geq y \wedge m,$ we get $(x \leftarrow y) \wedge m \geq (((x \leftarrow y) \leftarrow y) \leftarrow y) \wedge m.$

On the other hand, since $(x \vee (x \leftarrow y)) \wedge m = ((x \leftarrow y) \vee x) \wedge m \geq y \wedge m,$ we get $x \wedge m \geq ((x \leftarrow y) \leftarrow y) \wedge m.$

Replace x by $x \leftarrow y$ in the above, we get $(x \leftarrow y) \wedge m \geq (((x \leftarrow y) \leftarrow y) \leftarrow y) \wedge m.$

Thus we get (iii). Now $((z \leftarrow x) \leftarrow (z \leftarrow y)) \wedge m = (((z \leftarrow x) \vee z) \leftarrow y) \wedge m \leq (x \leftarrow y) \wedge m.$ To prove (v), since $(x \vee (y \leftarrow z) \vee (x \leftarrow y)) \wedge m = (x \vee y \vee (y \leftarrow z)) \wedge m = (x \vee y \vee z) \wedge m \geq z \wedge m,$

We get $(x \leftarrow y) \wedge m \geq ((y \leftarrow z) \leftarrow (x \leftarrow z)) \wedge m.$ Since $((x \vee y) \vee x \vee y \vee (x \leftarrow z)] \wedge [x \vee y \vee (y \leftarrow z)]\} \wedge m$ $(x \leftarrow z) \wedge (y \leftarrow z)) \wedge m = \{[=[(x \vee y \vee z) \wedge (x \vee y \vee z)] \wedge m \geq z \wedge m,$ we get $(x \leftarrow z) \wedge (y \leftarrow z) \wedge m \geq ((x \vee y) \leftarrow z) \wedge m.$

Thus we get (vi). Finally, since $(z \vee ((z \leftarrow x) \wedge (z \leftarrow y))) \wedge m = \{z \vee (z \leftarrow x)\} \wedge \{z \vee (z \leftarrow y)\} \wedge m \geq (x \wedge y) \wedge m,$ we get $(z \leftarrow x) \wedge (z \leftarrow y) \wedge m \geq (z \leftarrow (x \wedge y)) \wedge m.$

Theorem 3.15. Let $x, y \in A$ and $b, c \in B.$ Then we have the following:
(i) $((x \vee b) \leftarrow y) \wedge m = ((b \wedge m)' \wedge (x \leftarrow y)) \wedge m$
(ii) $(x \leftarrow (c \wedge y)) \wedge m = (c \wedge (x \leftarrow y)) \wedge m$
(iii) $((x \vee b) \leftarrow (c \wedge y)) \wedge m = ((b \wedge m)' \wedge c \wedge (x \leftarrow y)) \wedge m$ where $(b \wedge m)'$ is the complemented of $b \wedge m$ in $[0, m].$

Proof. Since $((x \vee b) \vee (b \wedge m)' \wedge (x \leftarrow y)) \wedge m = m \wedge (x \vee b \vee (x \leftarrow y)) \wedge m = (b \vee x \vee y) \wedge m \geq y \wedge m,$ we get $((b \wedge m)' \wedge (x \leftarrow y)) \wedge m \geq ((x \vee b) \leftarrow y) \wedge m.$ On the other hand, since $((x \vee b) \vee (x \vee b) \leftarrow y) \wedge m \geq y \wedge m,$ we get $(b \vee ((x \vee b) \leftarrow y)) \wedge m \geq (x \leftarrow y) \wedge m$ and hence $((x \vee b) \leftarrow y) \wedge m \geq (b \wedge m)' \wedge (x \leftarrow y) \wedge m.$ Thus we get (i). Similarly, we get (ii). Combining (i) and (ii) we get (iii).

In our paper, we introduce the concept of a dual Pseudo-Complemented Almost Distributive Lattices (or, simply a dual PCADL) [5] and studied its properties as follows.

1) **Definition 3.16.** [5] Let $(A, \vee, \wedge, 0$ be an ADL with a maximal element $m.$ Then, a unary operation $x \mapsto x_*$ on A is called a dual pseudo-complementation on A if, for any $x, y \in A,$ it satisfies the following conditions:
(i) $x \vee y = m \Rightarrow (x_* \vee y) \wedge m = y \wedge m$
(ii) $x \vee x_*$ is a maximal element in A
(iii) $(x \wedge y)_* = x_* \vee y_*.$

Now we prove the following theorem.

Theorem 3.17. A is a dual Pseudo-Complemented Almost Distributive Lattice.

Proof. Let A be a dual H-ADL with a maximal element m and $x \in A$. Define $x_* = x \leftarrow m$. Let $x, y \in A$. Suppose $x \vee y = m$. Then $(x_* \vee y) \wedge m = ((x \leftarrow m) \vee y) \wedge m = [x \leftarrow (x \vee y) \vee y] \wedge m = ((x \leftarrow y) \vee y) \wedge m = y \wedge m$. Now $(x \vee x_*) \wedge m = [x \vee (x \leftarrow m)] \wedge m = (x \vee m) \wedge m = m$. Thus $x \vee x_*$ is a maximal element in A. Also $(x \wedge y)_* = (x \wedge y) \leftarrow m = (x \leftarrow m) \vee (y \leftarrow m) = x_* \vee y_*$. Therefore A is a dual PCADL.

Finally, we conclude this paper with the following.

Theorem 3.18. Let A be an ADL with a maximal element m. Then A is a dual H-ADL if and only if PI(A) is a dual Heyting algebra.

Proof. Suppose A is a dual H-ADL. For $x, y \in A$, define $(x] \leftarrow (y] = (x \leftarrow y]$. Now we show $\leftarrow$ is well defined. Suppose $(x] = (x_1]$ and $(y] = (y_1]$. Then $x_1 \wedge x = x$, $x \wedge x_1 = x_1$ and $y_1 \wedge y = y$, $y \wedge y_1 = y_1$.

Now $(x \leftarrow y) \wedge m = (x \leftarrow (y \vee y_1)) \wedge m = ((x \leftarrow y) \vee (x \leftarrow y_1)) \wedge m = ((x \leftarrow y) \vee ((x_1 \wedge x) \leftarrow y_1))) \wedge m = [(x \leftarrow y) \vee (x_1 \leftarrow y_1) \vee (x \leftarrow y_1)] \wedge m \geq (x_1 \leftarrow y_1) \wedge m$. Then $(x_1 \leftarrow y_1] \subseteq (x \leftarrow y]$.

By symmetry, $(x \leftarrow y] \subseteq (x_1 \leftarrow y_1]$. Hence $(x \leftarrow y] = (x_1 \leftarrow y_1]$. It can be verified routinely, that $(PI(A), \vee, \wedge, \leftarrow, 0, A)$ is a dual Heyting algebra. Conversely, suppose that $PI(A)$ is a dual Heyting algebra. For $x, y \in A$, we define $x \leftarrow y = c \wedge m$ where $(x] \leftarrow (y] = (c]$. So that $(x] \leftarrow (y] = (c] = (c \wedge m] = (x \leftarrow y]$. If $(s] = (t]$, then $s \wedge m = t \wedge m$ and $\leftarrow$ is well defined. Let $x, y, z \in A$. Since $0/(x] \leftarrow (x] = (0$, we get $x \leftarrow x = 0 \wedge m = 0$. Now $(y] = \{(x] \leftarrow (y]\} \vee (y] = ((x \leftarrow y) \vee y]$ and hence $((x \leftarrow y) \vee y) \wedge m = y \wedge m$. Again, $(x \vee y] = (x] \vee (y] = \{(x] \vee \{(x] \leftarrow (y]\} = (x] \vee (x \leftarrow y] = (x \vee (x \leftarrow y)]$.

Therefore $(x \vee (x \leftarrow y)) \wedge m = (x \vee y) \wedge m$. Let $(z] \leftarrow (x] = (s]$ and $(z] \leftarrow (y] = (t]$. Then $z \leftarrow x = s \wedge m$ and $z \leftarrow y = t \wedge m$. So that $(z \leftarrow (x \vee y)] = (z] \leftarrow (x \vee y] = (z] \leftarrow \{(x] \vee (y]\} = (z] \leftarrow (x] \vee (z] \leftarrow (y]$ $= (z \leftarrow x] \vee (z \leftarrow y]$. Thus $(z \leftarrow (x \vee y)) \wedge m = ((z \leftarrow x) \vee (z \leftarrow y)) \wedge m$ and hence $z \leftarrow (x \vee y) = (z \leftarrow x) \vee (z \leftarrow y)$.

Similarly, we prove $(x \wedge y) \leftarrow z = (x \leftarrow z) \vee (y \leftarrow z)$. Thus A is a dual H-ADL.

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