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COMMON FIXED POINT THEOREMS FOR THREE SELF MAPS SATISFYING A GENERALIZED CONTRACTIVE CONDITION IN A CONE METRIC SPACE

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ABSTRACT

The aim of this paper is to present coincidence point and common fixed point theorems for three mappings satisfying generalized contractive conditions in a cone metric space. Our results generalize and extend some recent results in [5] and [6].

Subject classification: *47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

In 2007, Huang and Zhang [3] have generalized the concept of a metric space (called cone metric space) replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3], [4] and the references mentioned there in). Stojan Radenovic [5] has obtained coincidence point result for two mappings in cone metric spaces, which satisfy new contractive conditions.

Recently, M. Rangamma and K. Prudhvi [6] extended coincidence point results in [5] for three maps which satisfy generalized contractive condition without exploiting the nature of continuity.

In this paper we generalize the results in [5] and [6] by relaxing the contractive conditions used there in.

In all that follows, E is a real Banach space. For the mappings $f, g: X \to X$, let $C(f, g)$ denote the set of coincidence points of f and g, that is $C(f,g) = \{z \in X / fz = gz\}$.

Definition 1.1: ([3]) Let E be a real Banach space and P be a subset of E. P is called a cone if
a) P is closed, non-empty and $P \neq \{0\}$;
b) $a, b \in R,\ a,b \geq 0,\ x, y \in P \Rightarrow ax+by \in P$;
c) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Definition 1.2: ([3]) Let P be a cone in a Banach space E. Define partial ordering $\leq$ with respect to P by $x \leq y$ *if* $y-x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$, while $x << y$ means $y-x \in intP$, where $intP$ denotes the interior of the set P.

Definition 1.3: ([3]) Let X be a non-empty set. Suppose that a map d: $X \times X \to E$ satisfies
(1.3.1) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(1.3.2) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(1.3.3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.4: ([3]) Let E be a real Banach space and $P \subset E$ be a cone. P is called normal if there exists $k > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq k \|y\|$.

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The least positive number k satisfying the above inequality is called the normal constant of P.

Example 1.5: ([3]) Let $E=R^2$, $P = \{(x, y)\in E$ such that $x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \rightarrow E$ be defined by $d(x, y)= (|x-y|, \alpha|x-y|)$,where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.6: ([3]) Let (X, d) be a cone metric space. We say that a sequence $\{x_n\}$ in X is
(i) a Cauchy sequence if for every c in E with $0 << c$, there is a positive integer N such that for all $n, m > N$, $d(x_n, x_m) << c$.
(ii) a convergent sequence, if for any $0 << c$ there is a N such that for all $n > N$, $d(x_n, x) << c$ for some fixed x in X. We denote this by $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 1.7: ([3]) Let $f, g: X\rightarrow X$. Then the pair (f, g) is said to be weakly compatible at $z \in X$ if $f(g(z))= g(f(z))$ with $f(z) = g(z)$.

Lemma 1.8: ([3]) Let (X, d) be a cone metric space and let P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X.
Then
(i) $\{x_n\}$ converges to x if and only if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
(ii) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

In [5], (Stojan Radenovic [5], Theorem 2.1) the following theorem for two self maps on a cone metric space is proved.

Theorem 1.9: ([5]) Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose that the commuting mappings $f, g: X \rightarrow X$ are such that for some constant $\lambda \in (0, 1)$ and for every $x, y\in X$,

$$|| d (fx, fy) || \leq \lambda (||d (gx, gy)||) \tag{1.9.1}$$

If the range of g contains the range of f and if g is continuous, then f and g have a unique common fixed point.

In [6], (Rangamma and Prudhvi [6], Theorem 2.1) the above theorem is extended to three self maps and without exploiting the nature of continuity.

Theorem 1.10: ([6]) Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose the self maps f, g *and* h on X satisfy the condition:

$$|| d (fx, gy) || \leq \lambda (||d (hx, hy)||) \tag{1.10.1}$$

for all $x, y\in X$ where $\lambda \in (0,1)$ is a constant.

If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then the maps f, g and h have a coincidence point p in X. Moreover if (f, h) and (g, h) are weakly compatible at p, then f, g and h have a unique common fixed point.

Note: If we replace y by x in (1.10.1) we get fx = gx, for all x in X, so that f = g. Thus Theorem 1.10 is a result for two maps only, but not for three maps.

In [5], (Stojan Radenovic [5], Theorem2.3) the following theorem for two self maps on a cone metric space is proved.

Theorem 1.11: ([5]) Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings f, $g: X \rightarrow X$ satisfy

$$||d(fx, fy) || \leq \lambda (||d(fx, gx) ||+||d(fy, gy) ||), \text{ for all } x, y \in X \ldots \tag{1.11.1}$$

where $\lambda \in (0,½)$

If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence. Moreover if f and g are weakly compatible then f and g have a unique common fixed point.

In [6], (Rangamma and Prudhvi [6], Theorem 2.3) the above theorem is extended to three self maps.

Theorem 1.12: ([6]) Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose the self maps f, g and h on X satisfy the contractive condition:

$$||d(fx, gy)|| \leq \lambda (||d(fx, hx)||+||d(gy, hy)||), \text{ for all } x, y \in X . \tag{1.12.1}$$

where $\lambda \in [0,½)$ is a constant.

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If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then the maps f, g and h have a coincidence point p in X. Moreover if (f, h) and (g, h) are weakly compatible at p, then f, g and h have a unique common fixed point.

In [5], (Stojan Radenovic [5], Theorem 2.3) the following theorem for two self maps on a cone metric space is proved.

Theorem 1.13: ([5]) Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings $f, g: X \to X$ satisfy

$$\| d(fx, fy) \| \le \lambda (\|d(fx, gy)\|+\|d(fy, gx)\|), \text{ for all } x, y \in X \tag{1.13.1}$$

where $\lambda \in (0,½)$

If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence. Moreover if f and g are weakly compatible then f and g have a unique common fixed point.

2. MAIN RESULTS

In this section, we show that Theorem 1.10 (and hence Theorem 1.9) and Theorem 1.12 hold good even if we remove the restriction of 'norm' in (1.10.1) and (1.12.1). Further, we extend Theorem 1.13 to three self maps and remove the restriction of 'norm' in (1.13.1).

We first state and prove our first Theorem, in which the 'norm' restriction in (1.10.1) is removed.

Theorem 2.1: Let (X, d) be a cone metric space. Suppose the self maps f and h on X satisfy the condition:

$$d(fx, fy) \le \lambda\, d(hx, hy) \text{ for all } x, y \in X \tag{2.1.1}$$

where $\lambda \in (0,1)$ is a constant.

If $f(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then the maps f and h have a coincidence point z in X. Moreover if (f, h) is weakly compatible at z then f and h have a unique common fixed point.

Proof: Suppose x_0 is a point of X and define the sequence $\{y_n\}$ in X inductively as follows:

$$y_n = fx_n = hx_{n+1} \quad n = 0,1,2. \dots.$$

Consider,

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\le \lambda\, d(hx_n, hx_{n+1}) \quad by\ (2.1.1) \\ &= \lambda\, d(y_{n-1}, y_n) \end{aligned}$$

Therefore for all n,

$$d(y_{n+1}, y_{n+2}) \le \lambda\, d(y_n, y_{n+1}) \le \lambda^2\, d(y_{n+1}, y_{n+2}) \le \dots \le \lambda^{n+1} d(y_0, y_1)$$

Now for any $m > n$

$$\begin{aligned} 0 \le d(y_n, y_m) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{m-1}, y_m) \\ &\le \lambda^n d(y_0, y_1) + \lambda^{n+1} d(y_0, y_1) + \dots + \lambda^{m-1} d(y_0, y_1) \\ &= (\lambda^n + \lambda^{n+1} + \dots \lambda^{m-1})\, d(y_0, y_1) \\ &\le (\lambda^n + \lambda^{n+1} + \dots)\, d(y_0, y_1) \\ &= \frac{\lambda^n}{1-\lambda} d(y_{0,} y_1) \end{aligned}$$

$$\therefore \quad 0 \le d(y_n, y_m) \le \frac{\lambda^n}{1-\lambda}\, d(y_0, y_1)$$

That is $d(y_n, y_m) \to 0$ as $n, m \to \infty$. (Since $0 < \lambda < 1$)

Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = hx_{n+1}$

Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete, There exists q in $h(X)$ such that $\{hx_n\} \to q$ as $n \to \infty$.

Consequently, we can find z in X such that $hz = q$.

We shall show that $hz = fz$

Note that $d(hz, fz) = d(q, fz)$

Let us estimate $d(hz, fz)$

By triangle inequality,

$$d(hz, fz) \le d(hz, hx_{n+2}) + d(hx_{n+2}, fz)$$

$$= d(q, hx_{n+2}) + d(fx_{n+1}, fz)$$

Now,

$$d(fz, fx_{n+1}) \le \lambda\, d(hz, hx_{n+1}) \text{ (by (2.1.1))}$$

$$= \lambda\, d(q, hx_{n+1}) \to 0 \text{ as } n \to \infty.$$

Therefore for large n, we get

$$d(hz, fz) \le d(q, hx_{n+2}) + \lambda\, d(q, hx_{n+1}) \to 0 \text{ as } n \to \infty$$

which leads to $d(hz, fz) = 0$ and hence

$$hz = q = fz \tag{2.1.2}$$

From (2.1.2) it follows that

$q = hz = fz$ so that z is a coincidence point of f and h.

Since (f, h) is weakly compatible at z, we have,

$$d(ffz, fz) \le \lambda\, d(hfz, hz)$$

$$= \lambda\, d(fhz, fz)$$

$$= \lambda\, d(ffz, fz)$$

$$\therefore d(ffz, fz) = 0$$

$$\therefore ffz = fz$$

$$\therefore fz = ffz = fhz = hfz.$$

$$\Rightarrow ffz = hfz = fz = q.$$

Therefore fz $(=q)$ is a common fixed point of f and h (2.1.3)

From (2.1.3) it follows that f and h have a common fixed point, namely q.

The uniqueness of the common fixed point follows from equation (2.1.1).

Let q_1 be another common fixed point of f and h.

Then, $d(q, q_1) = d(fq, fq_1)$

$$\le \lambda\, d(hq, hq_1) \text{ (by (2.1.1))}$$

$$= \lambda\, d(q, q_1)$$

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As $0 < \lambda < 1$, it follows that $d(q, q_1) = 0$ that is $q = q_1$.

Therefore *f and h* have a unique common fixed point.

Now we state and prove our second theorem in which the "norm" restriction in (1.12.1) is removed.

Theorem 2.2: Let (X, d) be a cone metric space. Suppose the self maps f, g and h on X satisfy the condition:

$$d(fx, gy) \leq \lambda\, (d(fx, hx) + d(gy\,, hy)), \text{ for all } x, y \in X. \qquad (2.2.1),$$

where $\lambda \in [0,½)$ is a constant.

If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then the maps f, g and h have a coincidence point z in X. Moreover if (f, h) and (g, h) are weakly compatible at z, then f, g and h have a unique common fixed point.

Proof: Suppose x_0 is a point of X and define the sequence $\{y_n\}$ in X inductively as follows:

$$y_{2n} = fx_{2n} = hx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = hx_{2n+2}, \; n = 0,1,2,\ldots$$

Consider,

$$d(y_{2n}\,, y_{2n+1}) = d(fx_{2n}\,, gx_{2n+1})$$

$$\leq \lambda\, (d(fx_{2n}\,, hx_{2n}) + d\,(gx_{2n+1}\,, hx_{2n+1}))$$

$$= \lambda\, (d(y_{2n}\,, y_{2n-1}) + d(y_{2n+1}\,, y_{2n}))$$

$$\Rightarrow (1-\lambda)\, d(y_{2n}\,, y_{2n+1}) \leq \lambda\, d(y_{2n}\,, y_{2n-1})$$

$$\Rightarrow d(y_{2n}\,, y_{2n+1}) \leq \frac{\lambda}{1-\lambda}\, d(y_{2n}\,, y_{2n-1})$$

$$\Rightarrow d(y_{2n}\,, y_{2n+1}) \leq \delta\, d(y_{2n}\,, y_{2n+1}), \text{ where } \delta = \frac{\lambda}{1-\lambda} < 1$$

Similarly, it can be shown that

$$d(y_{2n+1}, y_{2n+2}) \leq \delta\, d(y_{2n}\,, y_{2n+1})$$

Therefore, for all n,

$$d(y_{n+1}\,, y_{n+2}) \leq \delta\, d(y_n, y_{n+1}) \leq \delta^2 d(y_{n-1}, y_n) \leq \ldots \leq \delta^{n+1} d(y_0, y_1)$$

Now, for any $m > n$

$$d(y_n\,, y_m) \leq d(y_n\,, y_{n+1}) + d(y_{n+1}\,, y_{n+2}) + \ldots + d(y_{m-1}\,, y_m)$$

$$\leq \delta^n d\,(y_0, y_1) + \delta^{n+1} d(y_0\,, y_1) + \ldots + \delta^{m-1} d(y_0\,, y_1)$$

$$\leq (\delta^n + \delta^{n+1} + \ldots)\, d(y_0\,, y_1)$$

$$= \frac{\delta^n}{1-\delta}\, d(y_0\,, y_1) \to 0 \text{ as } n, m \to \infty.$$

$\therefore$ $\{y_n\}$ is a Cauchy sequence, where $y_n = hx_{n+1}$

Therefore $\{hx_n\}$ is a Cauchy sequence.

Since $h(X)$ is complete, there exists q in $h(X)$ such that $hx_n \to q$ as $n \to \infty$. Consequently, we can find z in X such that $hz = q$.

We shall show that $hz = fz = gz$.

Consider, $d(fz\,, gx_{2n+1}) \leq \lambda\, (d\,(fz, hz) + d\,(gx_{2n+1}\,, hx_{2n+1}))$

On letting $n \to \infty$, we get

$d\,(fz\,,q) \;\leq\; \lambda\,(d(fz\,,hz) + d(q\,,q))$

$\quad = \lambda\, d\,(fz\,,q)$

$\Rightarrow (1-\lambda)\, d(fz\,,q) \leq\; 0$

$\therefore\; d(fz, q) \;\leq\; 0$

$\therefore\; fz = q = hz$ (2.2.2)

Similarly, we get $gz = q = hz$ (2.2.3)

By (2.2.2) and (2.2.3),

$q = hz = fz = gz,\quad z$ is a coincident point of f, g and h (2.2.4)

Since (f, h) and (g, h) are weakly compatible at z, we get, by (2.2.4) and contractive condition

$d(ffz, fz) = d(ffz, gz)$

$\quad \leq\; \lambda\,(d(\,ffz, hfz\,) + d(\,gz, hz))$

$\quad \leq\; \lambda\;\; (d(\,ffz, fhz\,)\;\; + d(gz, gz))\qquad (since\ hz = gz\,)$

$\quad = \lambda\, d\,(ffz, fhz)$

$\quad = \lambda\, d(fhz, fhz)\qquad (since\ fz = hz)$

$\quad = 0$

$\therefore\; d\,(ffz,fz)\;\leq\; 0.$

$\therefore\; ffz = \; fz$

$fz = \; ffz \;=\; fhz \;=\; hfz$

$\Rightarrow ffz = \; hfz \;=\; fz = \; q.$

Therefore fz $(=q)$ is a common fixed point of f *and* h (2.2.5)

Similarly, we get $gz = ggz = ghz = hgz$

$\Rightarrow ggz = hgz = gz = q$

$\therefore\; gz = \; (fz)\; (=q)$ is a common fixed point of g and h (2.2.6)

By (2.2.5) and (2.2.6), it follows that $f; g$ and h have a common fixed point q. The uniqueness of the common fixed point follows from equation (2.2.1). Let q_1 be another common fixed point of f, g and h

Consider,
$d(q, q_1) = \; d\,(fq, gq_1)$

$\quad \leq \lambda\,(\,d(fq, hq) + d(gq_1, hq_1)\,)$

$\quad = \lambda\,(\,d(hq, hq) + d(hq_1, hq_1)\,)$

$\quad = 0$

$\therefore\; d(q,q_1) \leq 0.$

Thus $q = q_1$

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Therefore f, g and h have a unique common fixed point.

Now we state and prove our third theorem in which the "norm" restriction in (1.13.1) is removed.

Theorem 2.3: Let (X, d) be a cone metric space. Suppose the self maps f, g and h on X satisfy the contractive condition:

$$d(fx, gy) \leq \lambda \left(d(fx, hy) + d(hx, gy) \right) \text{ for all } x, y \in X \tag{2.3.1}$$

where $\lambda \in [0,½)$ is a constant.

If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then the maps f, g and h have a coincidence point z in X. Moreover If (f, h) and (g, h) are weakly compatible at z, then f, g and h have a unique common fixed point.

Proof: Suppose x_0 is a point of X and define the sequence $\{y_n\}$ in X inductively as follows:

$$y_{2n} = fx_{2n} = hx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = hx_{2n+2}, \ n = 0,1,2,\ldots$$

Consider,

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq \lambda \left(d(fx_{2n}, hx_{2n+1}) + d(hx_{2n+1}, gx_{2n+1}) \right) \\ &= \lambda \left(d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1}) \right) \\ &= \lambda\, d(y_{2n-1}, y_{2n+1}) \\ &\leq \lambda \left(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) \right) \\ &= \lambda\, d(y_{2n-1}, y_{2n}) + \lambda\, d(y_{2n}, y_{2n+1}) \end{aligned}$$

$$\Rightarrow (1-\lambda)\, d(y_{2n}, y_{2n+1}) \leq \lambda\, d(y_{2n-1}, y_{2n})$$

$$\Rightarrow d(y_{2n}, y_{2n+1}) \leq \frac{\lambda}{1-\lambda}\, d(y_{2n-1}, y_{2n})$$

$$\therefore\ d(y_{2n}, y_{2n+1}) \leq \delta\, d(y_{2n-1}, y_{2n}), \text{ where } \delta = \frac{\lambda}{1-\lambda} < 1$$

Similarly it can be shown that

$$d(y_{2n+1}, y_{2n+2}) \leq \delta\, d(y_{2n}, y_{2n+1}).$$

Therefore, for all n,

$$d(y_{n+1}, y_{n+2}) \leq \delta\, d(y_n, y_{n+1}) \leq \delta^2\, d(y_{n-1}, y_n) \leq \ldots \leq \delta^{n+1} d(y_0, y_1)$$

Now, for any $m > n$

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m) \\ &\leq \delta^n d(y_0, y_1) + \delta^{n+1} d(y_0, y_1) + \ldots + \delta^{m-1} d(y_0, y_1) \\ &\leq (\delta^n + \delta^{n+1} + \ldots)\, d(y_0, y_1) \\ &= \frac{\delta^n}{1-\delta}\, d(y_0, y_1) \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

$\therefore$ $\{y_n\}$ is a Cauchy sequence, where $y_n = hx_{n+1}$

Therefore, $\{hx_n\}$ is a Cauchy sequence.

Since $h(X)$ is compete, there exists q in $h(X)$ such that $hx_n \to q$ as $n \to \infty$.

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Consequently we can find z in X such that $hz = q$.

We shall show that $hz = fz = gz$.

Consider, $d(fz, gx_{2n+1}) \leq \lambda\,(d(fz, hx_{2n+1}) + d(hz, gx_{2n+1}))$

Letting $n \to \infty$, we get

$d(fz, q) \leq \lambda\,(d(fz, q) + d(hz, q))$

$=\lambda\,(d(fz, q) + d(q, q))$

$\Rightarrow (1-\lambda)\, d(fz, q) \leq 0$

$\therefore d(fz, q) = 0$

$\therefore fz = q = hz$ (2.3.2)

Similarly, we get $gz = q = hz$ (2.3.3)

By (2.3.2) and (2.3.3)

$q = hz = fz = gz$, so that z is a coincidence point of f, g and h.

Since (f, h) and (g, h) are weakly compatible at z, we have,

$d(ffz, fz) = d(ffz, gz)$

$\leq \lambda\,(d(ffz, hz) + d(hfz, gz))$

$=\lambda\,(d(ffz, fz) + d(fhz, gz))$

$=\lambda\,(d(ffz, fz) + d(ffz, fz))$

$= \lambda\,(2d(ffz, fz))$

$\therefore d(ffz, fz) \leq 0$

$\therefore ffz = fz$

$\therefore fz = ffz = fhz = hfz$

$\Rightarrow ffz = hfz = fz = q$.

Therefore, $fz\ (=q)$ is a common fixed point of f and h (2.3.4)

Similarly, we get $gz = ggz = ghz = hgz$

$\Rightarrow ggz = hgz = gz = q.$

Therefore, $gz = (fz)\ (=q)$ is a common fixed point of g and h (2.3.5)

By (2.3.4) and (2.3.5), f, g and h have a common fixed point, namely, q. The uniqueness of the common fixed point of q follows from (2.3.1).

Let q_1 be another common fixed point of f, g and h.

Consider,

$d(q, q_1) = d(fq, gq_1)$

$\leq \lambda\,(d(fq, hq_1) + d(hq, gq_1))$

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$= \lambda\, (d(hq, hq_1) + d\, (hq, hq_1)\,)$

$= \lambda\, (2\, d(hq, hq_1)\,)$

$= \lambda\, (2\, d(q, q_1))$

$\therefore\, d(q, q_1) \,\leq\, 0.$

Thus $q = q_1$.

Therefore *f*, *g* and *h* have a unique common fixed point.

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