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A Coincidence point theorem for two hybrid pairs with quasi-contraction in b-metric spaces

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ABSTRACT

In this article, we give a coincidence point theorem for set-valued quasi-contraction maps in b-metric spaces.

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Keywords: *hybrid pairs, quasi contractions, b-metric space.*

1. INTRODUCTION

The study of fixed points for multi-valued contraction maps using the Hausdorff metric was initiated by Nadler [11] in 1969, who extended the Banach contraction principle to set-valued mappings. The theory of set-valued maps has many applications in control theory, convex optimization, differential equations and economics. Hassen Aydi, Monica-Felicia Bota, Erdal Karapinar and Slobodanka Mitrovic [10] proved a fixed point theorem for set-valued quasi-contractions in b-metric spaces. In this article we give a coincidence point theorem for set-valued quasi-contractions in b-metric spaces.

Definition 1.1. Let X be any nonempty set. An element x in X is said to be a fixed point of a multi-valued mapping T: $X \to 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X.

If T: $X \to 2^X$ is a multi-valued mapping and f : X→X is a self mapping then x is said to be a coincidence point of T and f if $fx \in Tx$.

Let (X, d) be a metric space. Let CB(X) be the collection of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, define

H (A, B) = max {δ (A, B), δ (B, A)},

where δ(A,B) = sup{d(a, B), $a \in A$}, δ(B,A)= sup{d(b, A), $b \in B$} with d(a, C) = inf{d(a, x), $x \in C$ }, $C \in CB(X)$

Then H is called the Hausdorff metric induced by the metric d.

Definition 1.2. [5] Let (X, d) be a metric space. The set-valued map T: X→X is said to be a q-set-valued quasi-contraction if there exists $0 \le q < 1$ such that for any $x, y \in X$,

$$d(Tx, Ty) \le q \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

Recently, Amini-Harandi [1] proved a theorem with a set-valued version of the above mentioned Ciric's condition [5] as follows:

Theorem 1.3. Let (X, d) be a complete metric-space. Suppose that T: X→CB(X) is a q-set-valued quasi-contraction. Assume that $q < \frac{1}{2}$. Then T has a fixed point in X, that is, there exists $u \in X$ such that $u \in Tu$.

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In the sequel, the letters R_+ and N will denote the set of all nonnegative real numbers and the set of all natural numbers respectively

Some problems, particularly the problem of the convergence of measurable functions with respect to a measure, lead to a generalization of notion of a metric. Using this idea, Czerwik [6] presented a generalization of the well known Banach fixed point theorem [2] in so-called b-metric spaces. Consistent with [6, 8], we use the following notations and definitions.

Definition 1.4. [6, 8] Let X be a nonempty set and $s \geq 1$ a given real number. A function d: XxX→R_+ is called a b-metric provided that, for all $x, y, z \in X$.
(bm-1) d(x, y) = 0 if and only if x = y,
(bm-2) d(x, y) = d(y, x),
(bm-3) d(x, y) ≤ s[d(x, z) + d(z, y)].

Note that a metric space is evidently a b-metric space. Czerwik [6, 8] has shown that a b-metric on X need not be a metric on X (see also [3, 4, 7, 9, 12]

.
We cite the following lemmas from Czerwik [6, 7, 8] and Singh *et. al.* [12].

Lemma 1.5. Let (X, d) be a b-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$. we have the following:
(i) d(x, B) ≤ d(x, b) for any $b \in B$,
(ii) δ(A, B) ≤ H(A, B),
(iii) d(x, B) ≤ H(A,B) for any $x \in A$.
(iv) H(A,A) = 0,
(v) H(A,B) = H(B,A),
(vi) H(A,C) ≤ s(H(A,B) + H(B,C)),
(vii) d(x, A) ≤ s(d(x, y) + d(y, A)).

Lemma 1.6. Let (X, d) be a b-metric space. Let A and B be in CB(X). Then for each α > 0 and for all $b \in B$ there exists $a \in A$ such that d(a, b) ≤ H(A,B) + α.

Lemma 1.7. Let (X, d) be a b-metric space. For $A \in CB(X)$ *and* $x \in X$. We have d(x, A) = 0 ⇔ $x \in \bar{A} = A$.

In this article, we establish the analogous of Theorem 1.3 on a complete b-metric space. The main theorem extends several well known comparable results in the existing literature.

Lemma 1.8. [13] Let (X, d) be a b-metric space and $\{y_n\}$ a sequence in X such that

$$d(y_{n+1},y_{n+2}) \leq \gamma\, d(y_n,y_{n+1}),\ n = 0,1\ldots$$

where $0 \leq \gamma < 1$. Then, $\{y_n\}$ is Cauchy sequence in X provided that $s\gamma < 1$.

Let (X, d) be a b-metric space. Again as in [1], the set-valued map T: X→CB(X) is said to be a q-set-valued quasi-contraction if for any $x, y \in X$

$$H(Tx, Ty) \leq q\, M(x, y),$$

where $0 \leq q < 1$ and M(x, y) = max{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)}.

2. MAIN RESULTS

Definition 2.1. Let (X,d) be a metric space. Let S, T: X→CB(X) be a set valued and f : X→X be a self map. The hybrid pairs (S, f) and (T, f) are said to be a q-set-valued quasi-contraction if there exists $0 \leq q < 1$ such that

$$H(Sx, Ty) \leq qM(x, y)$$

for all $x, y \in X$, where M(x, y) = max {d(fx, fy), d(fx, Sx), d(fy, Ty), d(fx, Ty), d(fy, Sx)}.

Theorem 2.2: Let (X, d) be a b-metric space. Suppose that f : X→X and S, T: X→CB(X). Suppose that the hybrid pairs (S, f) and (T, f) are q-set valued quasi-contraction, $\underset{x\in X}{\cup} Sx \subseteq f(X)$), $\underset{x\in X}{\cup} Tx \subseteq f(X)$ and f (X) is complete. Assume that $q < \dfrac{1}{s^2+s}$.

Then

(i) f and S ; f and T have coincidence points in X (or)

(ii) The pairs (S, f) and (T, f) have a common coincidence point.

Proof: Suppose M(x, y) = 0 for some x, y in X.

Then x is a coincidence point of f and S and y is a coincidence point of f and T.

Now assume that M(x, y) > 0 for all x, y in X.

Take $\varepsilon = \dfrac{1}{2}\left(\dfrac{1}{s^2+s} - q\right)$ *and* $\beta = q + \varepsilon = \dfrac{1}{2}\left(\dfrac{1}{s^2+s} + q\right)$.

Since $q < \dfrac{1}{s^2+s}$, we have $\varepsilon > 0$ and $0 < \beta < 1$.

Let $x_0 \in X$. Since $\underset{x\in X}{\cup} Sx \subseteq f(X)$ there exist $x_1 \in X$ and $y_1 \in X$ such that $y_1 = fx_1 \in Sx_0$.

Since $\underset{x\in X}{\cup} Tx \subseteq f(X)$ there exist $x_2 \in X$ and $y_2 \in X$. By Lemma 1.6, there exists $y_2 = fx_2 \in Tx_1$ such that

$$\begin{aligned} d(y_1, y_2) &\leq H(Sx_0, Tx_1) + \varepsilon M(x_0, x_1) \\ &\leq q M(x_0, x_1) + \varepsilon M(x_0, x_1) \\ &\leq \beta M(x_0, x_1). \end{aligned}$$

Similarly, there exists $y_3 = fx_3 \in Sx_2$ such that

$$d(y_2, y_3) \leq \beta M(x_1, x_2).$$

Thus by induction there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n+1} = fx_{2n+1} \in Sx_{2n} \ , y_{2n+2} = fx_{2n+2} \in Tx_{2n+1} \ \text{ and } d(y_{2n+1}, y_{2n+2}) \leq \beta M(x_{2n}, x_{2n+1}) \tag{2.1}$$

For simplicity, write $d_{2n+1} = d(y_{2n+1}, y_{2n+2})$.

From (2.1) we have

$$\begin{aligned} d_{2n+1} &\leq \beta\, M(x_{2n}, x_{2n+1}) \\ &= \beta \max \{d(fx_{2n}, fx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(fx_{2n+1}, Tx_{2n+1}), d(fx_{2n}, Tx_{2n+1}), d(fx_{2n+1}, Sx_{2n})\} \\ &\leq \beta \max \{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), 0\} \\ &\leq \beta \max \{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), s[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]\} \\ &= \beta \max \{d_{2n}, d_{2n+1}, s(d_{2n} + d_{2n+1})\} \end{aligned}$$

Thus $d_{2n+1} \leq \beta \max \{d_{2n}, d_{2n+1}, s(d_{2n} + d_{2n+1})\}$ (2.2)

If $\max \{d_{2n}, d_{2n+1}, s(d_{2n} + d_{2n+1})\} = d_{2n+1}$ then from (2.2) $0 < d_{2n+1} \leq \beta\, d_{2n+1}$, which is a contradiction since $0 < \beta < 1$.

Therefore, (2.2) becomes $d_{2n+1} \leq \beta \max\{d_{2n}, s(d_{2n}+d_{2n+1})\}$.

Put $\gamma = \max\left\{\beta, \dfrac{s\beta}{1-s\beta}\right\}$

Thus $d_{2n+1} \leq \gamma d_{2n}$, for all $n \in N$ (2.3)

If $\gamma = \beta$ then $s\gamma = s\left[\dfrac{1}{2(s^2+s)} + \dfrac{q}{2}\right] = s\left[\dfrac{1+q(s^2+s)}{2(s^2+s)}\right] < s\left[\dfrac{1+1}{2(s^2+s)}\right] = \dfrac{1}{1+s} \leq \dfrac{1}{2} < 1$.

If $\gamma = \dfrac{s\beta}{1-s\beta}$ then $s\gamma = s\left[\dfrac{s\beta}{1-s\beta}\right] < s\left[\dfrac{\frac{1}{1+s}}{1-\frac{1}{1+s}}\right] = 1$.

Thus $\gamma s < 1$.

From (2.3) we have $d(y_{2n+1}, y_{2n+2}) \leq \gamma\, d(y_{2n}, y_{2n+1})$ (2.4)

Similarly $d(y_{2n}, y_{2n+1}) \leq \gamma\, d(y_{2n-1}, y_{2n})$ (2.5)

From (2.4) and (2.5) we have

$$
\begin{aligned}
d(y_n, y_{n+1}) &\leq \gamma\, d(y_{n-1}, y_n) \\
&\leq \gamma^2 d(y_n, y_{n-1}) \\
&\;\;\vdots \\
d(y_n, y_{n+1}) &\leq \gamma^n d(y_0, y_1)
\end{aligned}
\tag{2.6}
$$

Now for $m, n \in N$ with $n < m$, we have

$$
\begin{aligned}
d(y_n, y_m) &\leq s d(y_n,y_{n+1}) + s^2 d(y_{n+1},y_{n+2}) + s^3 d(y_{n+2},y_{n+3}) + \ldots + s^{m-n} d(y_{m-1},y_m) \\
&\leq \gamma^n s d(y_0, y_1) + \gamma^{n+1} s^2 d(y_0, y_1) + \ldots + \gamma^{m-1} s^{m-n} d(y_0, y_1) \quad \text{from (2.6)} \\
&= \gamma^n s(1 + \gamma s + \gamma^2 s^2 + \ldots)\, d(y_0, y_1) \\
&\leq \frac{\gamma^n s}{1-\gamma s} d(y_0, y_1) \quad \to 0 \text{ as } n, m \to \infty.
\end{aligned}
$$

Hence $\{y_n\}$ is a Cauchy sequence in the b-metric space X. Since $f(X)$ is complete, there exists $z \in X$ such that $z = fu$ for some $u \in X$ such that

$\lim\limits_{n\to\infty} d(fx_{2n}, fu) = 0$ and $\lim\limits_{n\to\infty} d(fx_{2n+1}, fu) = 0$.

We have $d(fx_{2n+1}, Tu) \leq H(Sx_{2n}, Tu) \leq q\, M(x_{2n}, u)$ (2.7)

where

$M(x_{2n}, u) = \max\{d(fx_{2n}, fu), d(fx_{2n}, Sx_{2n}), d(fu, Tu), d(fx_{2n}, Tu), d(fu, Sx_{2n})\}$

We have $d(fx_{2n}, fu) \to 0$ as $n \to \infty$.

d(fx_{2n} ,Sx_{2n}) ≤ d(fx_{2n} , fx_{2n+1}) implies that d(fx_{2n} ,Sx_{2n}) →0 as n→∞.

Also d (fu , Sx_{2n}) ≤ d(fu , fx_{2n+1}) implies that d(fu , Sx_{2n}) →0 as n→∞.

Making n→∞ in (2.7) we get d (fu , Tu) ≤ q d (fu , Tu)

This gives $fu \in Tu$, since 0 ≤ q < 1. Hence u is coincidence point of T and f .

We have d(Su , fx_{2n+2}) ≤ H(Su ,Tx_{2n+1}) ≤ q M(u , x_{2n+1}) (2.8)

where

M(x_{2n}, u) = max{d(fu , fx_{2n+1}), d(fu , Su), d(fx_{2n+1} ,Tx_{2n+1}), d(fu , Tx_{2n+1}),d(fx_{2n+1} , Su)}.

We have d(fu , fx_{2n+1}) →0 as n→∞.

d (fx_{2n+1} ,Tx_{2n+1}) ≤ d(fx_{2n+1} , fx_{2n+2}) implies that d(fx_{2n+1} ,Tx_{2n+1}) →0 as n→∞.

Also d (fu , Tx_{2n+1}) ≤ d (fu , fx_{2n+2}) implies that d (fu , Tx_{2n+1}) →0 as n→∞.

Making n→∞ in (2.8) we get d (fu , Su) ≤ q d(fu , Su).

This gives $fu \in Su$, since 0 ≤ q < 1. Hence u is coincidence point of S and f

Thus u is a common coincidence point of f , S and T.

Corollary 2.3. Let (X, d) be a b-metric space. Suppose that f : X→X and S: X→CB(X). Suppose that the hybrid pair (S, f) is q-set valued quasi-contraction, $\underset{x\in X}{\cup} Sx \subseteq f(X)$ and f (X) is complete. Assume that $q < \dfrac{1}{s^2+s}$. Then S and f have a coincidence point.

Proof: Putting S = T in Theorem 2.2, we have S and f have a coincidence point. Now we give an example to illustrate Corollary 2.3.

Example 2.4: Let X= [0, ∞) be a complete b-metric space with d(x, y) = $|x-y|^2$ for all x, y ϵ X and s = 2. Define $fx = \dfrac{x}{2}$ and $Sx = \left[1, 1+\dfrac{x}{6}\right]$ for all x. Then $\underset{x\in X}{\cup} Sx \subseteq f(X).$

For all x, y in X, we have H(Sx, Sy) = $\left|\dfrac{x}{6}-\dfrac{y}{6}\right|^2 = \dfrac{1}{36}|x-y|^2 = \dfrac{1}{9}\dfrac{1}{4}|x-y|^2 = \dfrac{1}{9} d(fx.fy).$

All the hypotheses of the Corollary 2.3 are satisfied. Clearly S and f have coincidence points.

REFERENCES

[1]. Amini Harandi. A: Fixed point theory for set-valued quasi-contraction maps in metric spaces. Appl Math Lett. 24, 1791–1794 (2011). doi:10.1016/j.aml.2011.04.033.

[2]. Banach. S: Sur les operations dans les ensembles abstraits et leur application aux equations integrals. Fund Math. 3,133–181 (1922).

[3]. Boriceanu. M: Fixed point theory for multivalued generalized contraction on a set with two b- metrics. Studia Univ Babes- Bolyai Math. LIV (3), 1–14 (2009).

[4]. Boriceanu. M: Strict fixed point theorems for multivalued operators in b-metric spaces. Int. J Modern Math. 4(2), 285–301(2009).

[5]. Ćirić. LB: A generalization of Banach's contraction principle. Proc Am Math Soc. 45, 267– 273 (1974).

[6]. Czerwik. S: Contraction mappings in b-metric spaces. Acta. Math. Inf orm.Univ. Ostraviensis. 1, 5–11 (1993).

[7]. Czerwik. S, Dlutek. K, Singh. SL: Round-off stability of iteration procedures for operators in b-metric spaces. J Nature Phys Sci. 11, 87–94 (1997).

[8]. Czerwik. S: Nonlinear set-valued contraction mappings in b-metric spaces. Atti Sem Mat Fis Univ Modena. 46(2), 263–276 (1998).

[9]. Czerwik. S, Dlutek. K, Singh. SL: Round-off stability of iteration procedures for set-valued operators in b-metric spaces. J Nature Phys Sci. 11, 87–94 (2007).

[10]. Hassen Aydi, Monica-Felicia Bota, Erdal Karapinar and Slobodanka Mitrovic, A fixed point theorem for set-valued quasi-contractions in b-metric spaces, Fixed point theorey and Applications 2012,2012:88.

[11]. Nadler. SB: Multivalued contraction mappings. Pac J. Math. 30, 475–488 (1969).

[12]. Singh. SL, Bhatnagar. C, Mishra. SN: Stability of iterative procedures for multivalued maps in metric spaces.Demonstratio Math. 38(4), 905–916 (2005).

[13]. Singh. SL, Czerwik. S, Krol. K, Singh. A: Coincidences and fixed points of hybrid contractions. Tamsui Oxford J Math Sci. 24(4), 401–416 (2008).

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