

THE INDEPENDENT TRANSVERSAL NEIGHBOURHOOD NUMBER OF A GRAPH

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ABSTRACT

A set S of vertices in a graph G is a neighbourhood set of G if $G = \bigcup_{v \in S} \langle N[v] \rangle$ where $\langle N[v] \rangle$ is the subgraph of G induced by v and all points adjacent to v . A neighbourhood set $S \subseteq V$ of a graph G is said to be an independent transversal neighbourhood, if S intersects every maximum independent set of G . The minimum cardinality of an independent transversal neighbourhood set of G is called the independent transversal neighbourhood number of G and is denoted by $\eta_{it}(G)$. In this paper we begin an investigation of this parameter.

Keywords: Neighbourhood set, Independent set, Independent transversal neighbourhood set.

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INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by Chartrand and Lesniak [1].

In a graph $G = (V, E)$, the open neighbourhood of a vertex $v \in V$ is $N(v) = \{x \in V : vx \in E\}$, the set of vertices adjacent to v . The closed neighbourhood is $N[v] = N(v) \cup \{v\}$. A clique in a graph G is a complete subgraph of G . The maximum order of clique in G is called the clique number and is denoted by $\omega(G)$ and clique of order $\omega(G)$ is called a maximum clique. The subgraph induced by a set $S \subseteq V$ is denoted by $\langle S \rangle$.

A set $D \subseteq V$ is a dominating set, if every vertex in $V - D$ is adjacent to a vertex in D and the minimum cardinality of a dominating set is called the dominating number of G and is denoted by $\gamma(G)$. A survey of advanced topics in domination is given in the book by Haynes *et.al* [3].

Let $\deg(v)$ be the degree of vertex v and as usual $\delta(G)$, the minimum degree and $\Delta(G)$, the maximum degree of a graph. $\alpha_0(G)$ is the minimum number of vertices in a vertex cover of G . $\beta_0(G)$ is the minimum number of vertices in a maximal independent set of vertex of G . we employ the notation $\lceil x \rceil$ to denote the smallest integer greater than or equal to x , and $\lfloor x \rfloor$ to denote the largest integer less than or equal to x .

A dominating set $S \subseteq V$ of a graph G is said to be an independent transversal dominating set, if S intersects every maximum independent set of G . The minimum cardinality of an independent transversal dominating set of G is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. This concept was introduced by Hamid in [2].

A set S of vertices in a graph G is a neighbourhood set of G if $G = \bigcup_{v \in S} \langle N[v] \rangle$ Where $\langle N[v] \rangle$ is a subgraph of G induced by v and all points adjacent to v . The neighbourhood number $\eta(G)$ of a graph G equals the minimum number of vertices in a neighbourhood set of G [4]. In this paper we introduced another basic neighbourhood parameter namely independent transversal neighbourhood number and initiate the study of this new neighbourhood parameter.

A neighbourhood set $S \subseteq V$ of a graph G is said to be an independent transversal neighbourhood, if S intersects every maximum independent set of G . The minimum cardinality of an independent transversal neighbourhood set of G and is called an independent transversal neighbourhood number and denoted by $\eta_{it}(G)$.

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2. RESULTS

The following results are immediate

Proposition: 1

(i): For any graph G , $\gamma(G) \leq \eta(G) \leq \eta_{it}(G)$.

(ii): For any graph G , $\gamma(G) \leq \gamma_{it}(G) \leq \eta_{it}(G)$.

Proposition: A [4] For any path P_p of order p , $\eta(P_p) = \left\lfloor \frac{p}{2} \right\rfloor$

Theorem: 2 For any path P_p of order P , $\eta_{it}(P_p) = \left\lfloor \frac{P}{2} \right\rfloor + 1$.

Proof: Let $P_p = (v_1, v_2, \dots, v_p)$. Then $S = \{V_{2i} : 1 \leq i \leq 2n\}$ is the n -set of P_p

Further, $\langle V - S \rangle = \left\lfloor \frac{P}{2} \right\rfloor k_1$, and hence every independent set in $V - S$, contains at most $\left\lfloor \frac{P}{2} \right\rfloor$ vertices. Now since

$\left\lfloor \frac{P}{2} \right\rfloor = \beta(P_p)$. It follows that $S \cup \{x\}$, where $x \in V - S$ is an independent transversal neighbourhood set of P_p .

Hence $\eta_{it}(P_p) = |S| + 1$.

$$= \frac{P}{2} + 1.$$

Since $\eta(P_p) = \left\lfloor \frac{P}{2} \right\rfloor$

Proposition B. [4]: For any cycle C_p , with $p \geq 4$, $\eta(C_p) = \left\lfloor \frac{P}{2} \right\rfloor$

Theorem 3: For any cycle C_p of order P , $\eta_{it}(C_p) = \left\lfloor \frac{P}{2} \right\rfloor + 1$.

Proof: Let $C_p = (v_1, v_2, \dots, v_p)$. Then we consider two cases.

Case (i): If P is odd then $S = \{v_1, v_3, v_5, \dots, v_{p-1}\}$

i.e., $S = \{v_{2i+1} : 0 \leq i \leq \frac{p-1}{2}\}$ is a n -set of C_p .

Now, since $\langle V - S \rangle = \left\lfloor \frac{P}{2} \right\rfloor k_1$ every independent set in $V - S$ contains at most $\frac{P}{2}$ vertices and hence $V - S$ contains β_0 -set.

Thus it follows that $S \cup \{u\}$ where $u \in V - S$ is an independent transversal neighbourhood set of C_p .

Hence $\eta_{it}(C_p) = \eta(C_p) + 1$

$$= \left\lfloor \frac{P}{2} \right\rfloor + 1$$

Case (ii): If P is even, then

$s = \left\{v_{2i} : 1 \leq i \leq \frac{P}{2}\right\}$ is a n -set of C_p , and $\langle V - S \rangle = \left(\frac{P}{2}\right) k_1$, hence every independence set in $V - S$ contains $\frac{P}{2}$ vertices so that $V - S$ contains β_0 -set.

Thus it follows that $S \cup \{u\}$ where $u \in V-S$ is an independent transversal neighbourhood set of C_p .

$$\begin{aligned} \text{hence } \eta_{it}(C_p) &= |S \cup \{u\}| \\ &= n(C_p) + 1 \\ &= \frac{p}{2} + 1. \end{aligned}$$

From case (i) and (ii), we have $\eta_{it}(C_p) = \left\lceil \frac{p}{2} \right\rceil + 1$.

Corollary 1: For any wheel W_p on p vertices ($P \neq 4$) $\eta_{it}(W_p) = \left\lceil \frac{p-1}{2} \right\rceil + 1$.

Proof: Clearly, $\eta_{it}(W_p) = \eta_{it}(C_{p-1})$.
 $= \left\lceil \frac{p-1}{2} \right\rceil + 1$.

Theorem 4: For any graph G , we have $\eta(G) \leq \eta_{it}(G) \leq \eta(G) + \delta(G)$.

Proof: Since an independent transversal neighbourhood set of G is a neighbourhood set, it follows that $\eta(G) \leq \eta_{it}(G)$.

Now let u be a vertex in G with $\deg(u) = \delta(G)$, and let S be a n -set in G . Then every maximum independent set of G contains a vertex of $N(u)$, so that $S \cup N[u]$ is an independent transversal neighbourhood set of G . Also, since S intersect $N[u]$ it follows that $|S \cup N(u)| \leq \eta(G) + \delta(G)$ and hence the right inequalities follows.

Theorem 5: If G is a disconnected graph, with components G_1, G_2, \dots, G_r , then

$$\eta_{it}(G) = \min_{1 \leq i \leq r} \left\{ \eta_{it}(G_i) + \sum_{j=1, j \neq i}^r \eta(G_j) \right\}.$$

Proof: Let $G = \bigcup_{i=1}^r G_i$

Suppose that N_1, N_2, \dots, N_r are the maximum neighbourhood sets of the graphs G_1, G_2, \dots, G_r respectively and S_1, S_2, \dots, S_r are the minimum independent transversal neighbourhood sets of G_1, G_2, \dots, G_r .

Let B_1, B_2, \dots, B_r be the maximum independent sets of G_1, G_2, \dots, G_r . Any independent transversal neighbourhood set $S_i, i = 1, \dots, r$ intersect the set $\bigcup_{i=1}^r B_i$.

Hence $S_1 \bigcup_{i=2}^r N_i, S_2 \bigcup_{i=1, i \neq 2}^r N_i, \dots, S_r \bigcup_{j=1, j \neq r}^{r-1} N_j$ are all the independent transversal neighbourhood of G and the order of those sets will be,

$$\begin{aligned} &\eta_{it}(G_1) + \sum_{i=2, j \neq 1}^r \eta(G_j) \\ &\eta_{it}(G_2) + \sum_{i=1, j \neq 2}^r \eta(G_j) \\ &\eta_{it}(G_3) + \sum_{i=1, j \neq 3}^r \eta(G_j) \quad \vdots \quad \vdots \\ &\eta_{it}(G_i) + \sum_{i=1, j \neq i}^r \eta(G_j) \end{aligned}$$

Since the minimum independent transversal neighbourhood set is one of the set $S_1 \bigcup_{i=1}^r N_i$ which has the minimum cardinality.

$$\text{Hence } \eta_{it}(G) = \min_{1 \leq i \leq r} \left\{ \eta_{it}(G_i) + \sum_{j=1, j \neq i}^r \eta(G_j) \right\}.$$

Theorem 6: For any non-complete graph G with clique number ω , $\eta_{it}(G) \leq \eta - \omega + 1$. Further equality holds if and only if $\beta_0(G) = 2$.

Proof: Let H be a maximum clique in G . Let $u \in V(H)$. Then $S = V(G) - V(H-u)$ is neighbourhood set of G . Since $\beta_0(G) \geq 2$ and H is a maximum clique, it follows that every maximum independent set of G intersects S . Hence S is an independent transversal neighbourhood set so that $\eta_{it}(G) \leq n - \omega + 1$.

Suppose $\eta_{it}(G) \leq \eta - \omega + 1$. Let H be a maximum clique in G . Let u and v be two adjacent vertices such that $u \in V(H)$ and $v \in V(G) - H$. Then $N = \{u\} \cup [V(G) - V(H \cup v)]$ is a neighbourhood set of G with $|N| = \eta - \omega$. Since $\eta_{it}(G) = n - \omega + 1$, there exists a β_0 - set in G such that $N \cap S = \emptyset$. Hence S consists of the vertex u and a vertex $w \neq u$ in H , so that $\beta_0(G) = 2$.

Theorem 7: Let l and m be two positive integers with $m \geq 2l - 1$. Then there exist a graph G on m vertices, such that $\eta_{it}(G) = l + 1$.

Proof: Let $m = 2l + r$, $r \geq -1$ and let H be any connected graph on l vertices.

Let $V(H) = \{v_1, v_2, \dots, v_l\}$. Let G be a graph obtained from H by attaching $r + 1$ pendant edges at V_1 and one edge at each V_i for $i \geq 2$. Let u_i ($i \geq 2$) be the pendent vertex in G adjacent to V_i clearly $\eta(G) = l$ and $S = \{v_1, v_2, \dots, v_l, u_i\}$ (u_i is any pendent vertex in G) is a η_{it} set of G . Further every maximal independent set of G intersect S and hence $\eta_{it}(G) = |S| = l + 1$, also $|V(G)| = b$.

Theorem 8: For any non-complete graph G with $\delta(G) \geq 2$, we have $\eta_{it}(G) \leq \alpha_0(G)$.

Proof: Let S be a β_0 set of G . Then $V - S$ is a neighbourhood set of G , since, $G \neq K_2$ and $\delta(G) \geq 2$, there exists a vertex v in $V - S$ such that $|N(v) \cap S| \geq 2$. Let u and w be two neighbours of v in S , since $\delta(G) \geq 2$, it follows that every neighbourhood of v in S is adjacent to at least one vertex other than v in $V - S$ and hence $D = (V - S) - \{v\}$ is a neighbourhood set of $G - \{v\}$.

Then $D \cup \{w\}$ is an independent transversal neighbourhood set of G . This is because $(S - \{w\}) \cup \{v\}$ is the only set in the compliment of $D \cup \{w\}$, which is not an independent set, and hence $\eta_{it}(G) \leq \eta - \beta_0(G) = \alpha_0(G)$.

Corollary 2: If G is a non-complete graph $\eta_{it}(G) = \alpha_0(G) + 1$, then $\eta(G) = \alpha_0(G)$.

Proof: Suppose $\eta_{it}(G) = \alpha_0(G) + 1$. It follows from theorem 8 and 4 that $\eta_{it}(G) \leq \eta(G) + 1$ and hence $\alpha_0(G) \leq \eta(G)$.

Also since it is always true that $\eta(G) \leq \alpha_0(G)$,

we have $\eta(G) = \alpha_0(G)$.

Theorem 9: For any graph G , $1 \leq \eta_{it}(G) \leq p$. Further $\eta_{it}(G) = p$, if and only if $G = K_p$ or $\overline{K_p}$.

Proof: The inequalities are trivial. Suppose $\eta_{it}(G) = p$. assume $G = K_p$ or $\overline{K_p}$. Then G has at least three vertices u, v and w such that u and v are adjacent and w is not adjacent to one of u, v .

Suppose w is not adjacent to u . This implies that $V - \{u\}$ is an independent transversal n -set of G and hence $\eta_{it}(G) \leq p - 1$, which is a contradiction. This proves necessity, sufficient is obvious. The following are some interesting open problem

(1) Characterize graphs for which

(i) $\gamma_{it}(G) = \eta_{it}(G)$

(ii) $\gamma(G) = \gamma_{it}(G) = \eta(G) = \eta_{it}(G)$.

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