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A FIXED POINT THEOREM USING SEMI COMPATIBLE MAPPINGS IN FUZZY METRIC SPACE

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ABSTRACT

The aim of this paper, we prove a common fixed point theorem for six mappings under the condition of semi compatible mappings in fuzzy metric spaces. All the results of this paper are new.

Keywords: *Fuzzy metric space, t-norm, compatible maps, semi compatible, common fixed point.*

2000 Mathematics Subject Classification: *47H10, 54H25.*

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [14]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduces by Kromosil and Michalek [9], and George and Veeramani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Recently, many authors have proved fixed point theorems involving fuzzy sets [1], [5], [7], [11]. Vasuki [13] and Singh and Chauhan [12] introduce the concept of R-weakly commuting and compatible maps, respectively, in fuzzy metric space. Recently, Cho *et al* [4] initiated the concept of compatible maps of type (β) in fuzzy metric spaces by giving interesting relationship of type of mapping with compatible and compatible of type (α) mappings. In [3], Cho, Sharma and Sahu introduced the non symmetrical concept of semi compatible of maps in d-complete topological spaces.

In this paper we prove common fixed point theorem for semi compatible maps in fuzzy metric space. Without assuming either the completeness of the space or continuity of the mappings involved. We begin with definitions and preliminary concepts.

2. PRELIMINARIES

Definition (2.1) [6]: A binary operation $*: [0, 1] \to [0, 1]$ is a continuous t -norm if $*$ is a satisfying the following conditions:

(a) $*$ is a commutative and associative;
(b) $*$ is a continuous;
(c) $a*1 = a$ for all $a \in [0, 1]$;
(d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ and $a, b, c, d \in [0, 1]$.

Definition (2.2) [6]: The triplet $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, 1] \to [0, 1]$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot): [0, 1] \to [0, 1]$ is continuous,
(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. we identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

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Example (2.3) [6]: Let (X, d) be a metric space Define a ∗ b = min {a, b} and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all x, y ∈ X and all t > 0, then (X, M, ∗) is a fuzzy metric space. It is called the fuzzy metric space induces by d.

Definition (2.4) [6]: Let (X, M, ∗) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be converges to a point x∈ X if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t > 0. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all t > 0 and p > 0. The space is said to be complete if every Cauchy sequence in X converges to a point in X.

Lemma (2.1) [5]: For all x, y∈ X, M (x, y, ·) is a non - decreasing function.

Definition (2.5) [10]: Two self maps A and B of fuzzy metric space (X, M, ∗) are said to be weakly commuting if $M(ABx, BAx, t) \geq M(Ax, Bx, t)$ for every x∈ X.

The notion of weak commutativity is extended to R-weak commutativity by Vasuki [8] as

Definition (2.6) [13]: Two self maps A and B of fuzzy metric space (X, M, ∗) are said to be R-weakly commuting provided there exists some positive real number R such that

$$M(ABx, BAx, t) \geq M(Ax, Bx, \tfrac{t}{R}) \text{ for all } x\in X.$$

The weak commutativity implies R-weak commutativity and converge is true for R≤ 1.

Definition (2.7) [12]: Let A and B be two self maps of a fuzzy metric space (X, M, ∗), then A and B are said to be compatible if $M(ABx_n, BAx_n, t) \to 1$ as $n \to \infty$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ as $n \to \infty$, for some z∈ X.

Definition (2.8) [8]: Let A and B be two self maps of a fuzzy metric space (X, M, ∗), then A and B are said to be weakly compatible if they commute at their coincidence point, i.e.

$$ABu = BAu \text{ whenever } Au = Bu,\ u\in X.$$

Definition (2.9) [12]: A pair (A, B) of self maps of a fuzzy metric space (X, M, ∗) is said to be semi-compatible if $\lim_{n\to\infty} ABx_n = Bx$ whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x\in X.$$

Remark (2.1) [12]: Let (A, B) be self mappings of a fuzzy metric space (X, M, ∗). Then (A, B) is R- commuting implies (A, B) is semi compatible but the converse is not true.

Example (2.2) [2]: Let X = [0, 2] and a ∗ b = min {a, b}. Let $M(x, y, t) = \frac{t}{t+d(x, y)}$ be the standard fuzzy metric space induced by d. where $d(x, y) = |x - y|$ for all x, y∈ X. define

$$A(X) = \begin{cases} 2, & x \in 0, 1] \\ \frac{x}{2}, & x \in (1, 2] \end{cases}, \qquad B(X) = \begin{cases} 1, & x \in [0, 1) \\ 2, & x = 1 \\ \frac{x+3}{5}, & x \in (1, 2] \end{cases}$$

Now for 1 < x ≤ 2 we have

$Ax = \frac{x}{2}$, $Bx = \frac{x+3}{5}$ and $ABx = \frac{x+3}{10}$, $BAx = \frac{x+6}{10}$

Then $M(ABx, BAx, t) = \frac{10t}{10t + 3}$. $M(Ax, Bx, \frac{t}{R}) = \frac{10t}{10t + 3(2-x)R}$.

We observe that $M(ABx, BAx, t) \geq M(Ax, Bx, \frac{t}{R})$ which gives $R \geq \frac{1}{(2-x)}$

Therefore we get there no R for x∈ (1, 2] in X.

Hence (A, B) is not R-weakly commuting.

Now we have B (1) = 2 = A (1) and B (2) = 1 = A (2) also BA (1) = AB (1) and AB (2) = 2 = BA (2).

Let $x_n = 2 - \frac{1}{2n}$. Hence $Ax_n \to 1$, $Bx_n \to 1$ and $ABx_n \to 2$

Therefore $M(ABx_n, By, t) = (2, 2, t) = 1$.

Hence (A, B) is semi compatible but not R-weakly commuting.

Lemma (2.2) [11]: Let (X, M, $*$) be a fuzzy metric space. If there exists $k \in (0, 1)$ such that

$$M(x, y, kt) \geq M(x, y, t) \text{ then } x = y.$$

Lemma (2.3) [4]: Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, $*$) with the condition (FM-6), If there exists $k \in (0, 1)$ such that $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $t > 0$ and $n \in N$, then $\{y_n\}$ is a Cauchy sequence in X.

3. MAIN RESULT

Theorem (3.1): Let A, B, S, T, P and Q are self maps on a fuzzy metric space (X, M, $*$) satisfying:
(3.1.1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$
(3.1.2) AB = BA, ST = TS, PB = BP, QT = TQ.
(3.1.3) Either AB or P is continuous.
(3.1.4) (P, AB) is semi compatible and (Q, ST) is weak compatible.
(3.1.5) There exists $k \in (0, 1)$ such that

$M(Px, Qy, kt) \geq \text{Min}\{M(ABx, Px, t), M(STy, Qy, t), M(STy, Px, \beta t), M(ABx, Qy, (2-\beta)t), M(ABx, STy, t)\}$ for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$.

Then self-maps A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Let $x_0 \in X$. From condition (3.1.1) there exists $x_1, x_2 \in X$ such that $Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2…

Now we prove $\{y_n\}$ is a Cauchy sequence in X.

Step I: putting $x = x_{2n}$, $y = x_{2n+1}$ for $x > 0$ and $\beta = 1 - q$ with $q \in (0, 1)$ in (3.1.5) we get,

$$M(Px_{2n}, Qx_{2n+1}, kt) \geq \text{Min}\{M(ABx_{2n}, Px_{2n}, t), M(ABx_{2n}, Qx_{2n+1}, t), M(STx_{2n+1}, Px_{2n}, (1+q)t), M(ABx_{2n}, Qx_{2n+1}, (1+q)t), M(ABx_{2n}, STx_{2n+1}, t)\}$$

$$M(y_{2n}, y_{2n+1}, kt) \geq \text{Min}\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), 1, M(y_{2n-1}, y_{2n+1}, (1+q)t), M(y_{2n-1}, y_{2n}, t)\}$$

$$\geq \text{Min}\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, qt)\}$$

$$\geq \text{Min}\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\}$$

As t–norm is continuous letting q = 1, we get

$$M(y_{2n}, y_{2n+1}, kt) \geq \text{Min}\{M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\}$$

$$\geq \text{Min}\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)\}$$

Hence, $M(y_{2n}, y_{2n+1}, kt) \geq \text{Min}\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)\}$

Similarly, $M(y_{2n+1}, y_{2n+2}, kt) \geq \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}$

Therefore, for all n even or odd we have,

$$M(y_n, y_{n+1}, kt) \geq \text{Min}\{M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t)\}$$

Consequently, $M(y_n, y_{n+1}, t) \geq \text{Min}\{M(y_{n-1}, y_n, k^{-1}t), M(y_n, y_{n+1}, k^{-1}t)\}$

By repeated application of above inequality we get,

$M(y_n, y_{n+1}, kt) \geq \text{Min}\{M(y_{n-1}, y_n, k^{-1}t), M(y_n, y_{n+1}, k^{-m}t)\}$

Since $M(y_n, y_{n+1}, k^{-m}t) \to 1$ as $n \to \infty$ it follows that

$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) \quad \forall\, n \in N$ and $\forall\, x > 0$,

Therefore, by lemma (2.3), $\{y_n\}$ is a Cauchy sequence in X, which is complete.

Hence, $\{y_n\} \to z \in X$. Also its subsequences

$\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$ (1)

$\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$ (2)

Case: I P is continuous
As P is continuous $P^2x_{2n} \to Pz$ and $P(ABx_{2n}) \to Pz$.

As the limit of a sequence in fuzzy metric space is unique we have,

$ABz = Pz$ (3)

Step I: putting $x = z$, $y = x_{2n+1}$ with $\beta = 1$ in condition (3.1.5) we get,

$M(Pz, Qx_{2n+1}, kt) \geq \text{Min}\{M(ABz, Pz, t), M(STx_{2n+1}, Qx_{2n+1}, t), M(STx_{2n+1}, Pz, t), M(ABz, Qx_{2n+1}, t),$
$M(ABz, STx_{2n+1}, t)\}$

Letting $n \to \infty$ and using (1) and (3) we get,

$M(Pz, z, kt) \geq \text{Min}\{M(z, Pz, t), M(z, z, t), M(z, Pz, t), M(Pz, z, t), M(Pz, z, t)\}$

$M(Pz, z, kt) \geq M(Pz, z, t)$

Which gives $Pz = z$. therefore $ABz = Pz = z$.

Step II: As $P(X) \subseteq ST(X)$ there exists $v \in X$ such that $z = Pz = STv$.

Putting $x = x_{2n}$, $y = v$ with $\beta = 1$ in condition (3.1.5) we get,

$M(Px_{2n}, Qv, kt) \geq \text{Min}\{M(ABx_{2n}, Px_{2n}, t), M(STv, Qv, t), M(STv, Px_{2n}, t), M(ABx_{2n}, Qv, t), M(ABx_{2n}, STv, t)\}$

Letting $n \to \infty$ using equation (2) we get,

$M(z, Qv, kt) \geq \text{Min}\{M(z, z, t), M(z, Qz, t), M(z, z, t), M(z, Qv, t), M(z, z, t)\}$

$\geq M(z, Qv, t)$

Therefore by lemma (2.2), $Qv = z$. Hence $STv = z = Qv$. As (Q, ST) is weak compatible

We have. $ST(Qv) = Q(STv)$. Thus $STz = Qz$.

Step III: putting $x = x_{2n}$, $y = z$ with $\beta = 1$ in condition (3.1.5) we get,

$M(Px_{2n}, Qz, kt) \geq \text{Min}\{M(ABx_{2n}, Px_{2n}, t), M(STz, Qz, t), M(STz, Px_{2n}, t), M(ABx_{2n}, Qz, t), M(ABx_{2n}, STz, t)\}$

Letting $n \to \infty$ using equation (2) we get,

$M(z, Qz, kt) \geq \text{Min}\{M(z, z, t), M(Qz, Qz, t), M(Qz, z, t), M(z, Qz, t), M(z, Qz, t)\}$

i.e. $M(z, Qz, kt) \geq M(z, Qz, t)$

Hence, $z = Qz$.

Step IV: putting $x = x_{2n}$, $y = Tz$ with $\beta = 1$ in condition (3.1.5) we get,

$$M(Px_{2n}, QTz, kt) \geq Min\{M(ABx_{2n}, Px_{2n}, t), M(STTz, Px_{2n}, t), M(STTz, Px_{2n}, t), M(ABx_{2n}, QTz, t), M(ABx_{2n}, STTz, t)\}$$

As QT = TQ and ST = TS we have QTz = TQz = Tz and ST(Tz) = T(STz) = Tz.

Letting $n \to \infty$ we get,

$$M(z, Tz, kt) \geq Min\{M(z, z, t), M(Tz, z, t), M(Tz, Tz, t), M(Tz, z, t), M(z, Tz, t), M(z, Tz, t)\}$$

$$\geq M(z, Tz, t)$$

Therefore by lemma (2.2), Tz = z.

Now STz = Tz = z implies Sz = z. Hence Sz = Tz = Qz = z. (4)

Step V: Putting $x = Bz$, $y = x_{2n+1}$ with $\beta = 1$ in condition (3.1.5) we get,

$$M(PBz, Qx_{2n+1}, kt) \geq Min\{M(ABBz, PBz, t), M(STx_{2n+1}, Qx_{2n+1}, t), M(STx_{2n+1}, PBz, t), M(ABBz, Qx_{2n+1}, t), M(ABBz, STx_{2n+1}, t)\}$$

As BP = PB, AB = BA so we have P(Bz) = B(Pz) = Bz and AB(Bz) = B(ABz) = Bz.

Letting $n \to \infty$ we have,

$$M(Bz, z, kt) \geq Min\{M(Bz, Bz, t), M(z, z, t), M(z, Bz, t), M(Bz, z, t), M(Bz, z, t)\}$$

i.e. $M(Bz, z, kt) \geq M(Bz, z, t)$

which gives Bz = z and ABz = z implies Az = z. Therefore,

Az = Bz = Pz = z (5)

Combining (4) and (5) we have Az = Bz = Pz = Sz = Tz = Mz = z.

i.e. z is the common fixed point of the six maps P, Q, A, B, S and T in this case.

Case :II AB is continuous

As AB is continuous $(AB)^2x_{2n} \to ABz$ and $(AB)Px_{2n} \to ABz$.

As (P, AB) is semi compatible we have, $P(AB)x_{2n} \to ABz$.

Step VI: Putting $x = ABx_{2n}$, $y = x_{2n+1}$ with $\beta = 1$ in condition (3.1.5) we get,

$$M(PABx_{2n}, Qx_{2n+1}, kt) \geq Min\{M(ABABx_{2n}, PABx_{2n}, t), M(STx_{2n+1}, Qx_{2n+1}, t), M(STx_{2n+1}, PABx_{2n}, t), M(ABABx_{2n}, Mx_{2n+1}, t), M(ABABx_{2n}, STx_{2n+1}, t)\}$$

Letting $n \to \infty$ we get,

$$M(ABz, z, kt) \geq Min\{M(ABz, ABz, t), M(z, z, t), M(z, ABz, t), M(ABz, z, t), M(ABz, z, t)\}$$

i.e. $M(ABz, z, kt) \geq M(ABz, z, t)$

Therefore, by lemma (2.2), ABz = z.

Step VII: Putting $x = z$, $y = x_{2n+1}$ with $\beta = 1$ in condition (3.1.5) we get,

$$M(Pz, Qx_{2n+1}, kt) \geq Min\{M(ABz, Pz, t), M(STx_{2n+1}, Qx_{2n+1}, t), M(STx_{2n+1}, Pz, t), M(ABz, Qx_{2n+1}, t), M(ABz, STx_{2n+1}, t)\}$$

Letting $n \to \infty$ we get,

$M(Pz, z, kt) \geq Min\{M(z, Pz, t), M(z, z, t), M(z, Pz, t), M(Pz, z, t), M(Pz, z, t)\}$

i.e. $M(Pz, z, kt) \geq M(Pz, z, t)$

which gives Pz = z, therefore ABz = Pz = z.

Now, apply step (VI), to get Bz = z and we get Az = Pz = z = Bz = z.

Now using step (II), (III) and (IV) of previous case we get Qz = STz = Sz = Tz = z.

i.e. z is the common fixed point of the six maps P, Q, A, B, S and T in this case also.

Uniqueness: Let u be another fixed point of A, B, S, T, P and Q then Au = Bu = Qu = Su = Tu = Pu = u.

Putting x = z, y = u in condition (3.1.5) with $\beta = 1$ we get,

$M(Pz, Qu, kt) \geq Min\{M(ABz, Pz, t), M(STu, Qu, t), M(STu, Pu, t), M(ABz, Qu, t), M(ABz, STu, t)\}$

i.e. $M(z, u, kt) \geq M(z, u, t)$

which gives z = u. Therefore, z is a unique common fixed point of A, B, S, T, P ad Q.

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