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NEIGHBOURHOOD CONNECTED 2-EQUITABLE DOMINATION IN GRAPHS

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ABSTRACT

Let G= (V, E) be a graph, two vertices u and v in V said to be equitable adjacent, if u and v are adjacent in G and |deg (u)–deg(v)|≤1. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called equitable domination number of G. In this paper we introduce the neighbourhood connected 2-equitable domination number in graph, exact value for some standard graphs bounds and some interesting results are obtained.

***Keywords**: Equitable domination number,2-equitable dominating set, Neighbourhood Connected 2-equitable dominate-on in Graphs, chromatic number.*

***Mathematics Subject Classification:** 05C69.*

1. INTRODUCTION

Introduction: By a graph $G=(V,E)$ we mean a finite, undirected with neither loops nor multiple edges the order and size of G are denoted by p and q respectively for graph theoretic terminology we refer to Chartrand and Lesnaik [2] A subset S of V is called a dominating set if $N[S]=V$ the minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$, $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes etal [5] A survey of several advanced topics in domination is given in the book edited by Haynes *et al.* [6]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes *et al.* [5]. Sampathkumar and Walikar [8] introduced the concept of connected domination in graphs. Let $G=(V,E)$ be a graph and let $v\in V$ the open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[v]=N(v)\cup v$ respectively. If $S\subseteq V$ then $N(S)=\cup_{v\in S}N(v)$ and $N[S]=N(S)\cup S$. If $S\subseteq V$ and $u\in S$ then the private neighbor set of u with respect to S is defined by $Pn[u,S]=\{v:N[v]\cap S=\{u\}\}$.

A dominating set S of G is called a connected dominating set if the induced subgraph $\langle S\rangle$ is connected the minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S)\rangle$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$. A ncd-set S is said to be minimal if no proper subset of S is a ncd-set. A coloring of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum integer K for which a graph G is $k-$colorable is called the chromatic number of G and is denoted by $\chi(G)$.

A subset S of V is called an equitable dominating set if for every $v\in V-S$ there exist a vertex $u\in S$ such that $uv\in E(G)$ and $|d(u)-d(v)|\le 1$. The minimum cardinality of such an equitable dominating set is denoted by γ_e and is called the equitable domination number of G. A vertex $u\in V$ is said to be degree equitable with a vertex $v\in V$ if $|d(u)-d(v)|\le 1$. If S is an equitable dominating set then any super set of S is an equitable dominating

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set. An equitable set S is said to be a minimal equitable dominating set if no proper subset of S is an equitable dominating set. The minimal upper equitable dominating number is Γ_e the upper equitable dominating set of G. If $u \in V$ such that $|d(u)-d(v)| \geq 2$ for every $v \in N(u)$ then u is in every equitable dominating set such points are called an equitable isolated. I_e denotes the set of all equitable isolates. An equitable dominating S of connected graph G is called an equitable connected dominating set (ecd-set) if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a ecd-set of G is called the equitable connected domination number of G and is denoted by $\gamma_{ec}(G)$. Let $G=(V,E)$ be a graph and let $u \in V$ the equitable neighborhood of u denoted by $N_e(u)$ is defined as $N_e(u)=\{v \in V : |v \in N(u), |d(u)-d(v)| \leq 1 \}$ The maximum and minimum equitable degree of a point in G are denoted by $\Delta_e(G)$ and $\delta_e(G)$ that is $\Delta_e(G)=max_{u \in V(G)} |N_e(u)|$ and $\delta_e(G)=min_{u \in V(G)} |N_e(u)|$. The open equitable neighbourhood and closed equitable neighbourhood of v are denoted by $N_e(v)$ and $N_e[v]=N_e(v) \cup \{v\}$ respectively. If $S \subseteq V$ then $N_e(S)=\cup_{v \in S} N_e(v)$ and $N[S]=N_e(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private equitable neighbor set of u with respect to S is defined by $pne[u,S]=N_e[u]-N_e[S-\{u\}]$.

If G is connected graph, then a vertex cut of G is a subset R of $V(G)$ with the property that the subgraph of G induced by $V(G)-R$ is disconnected. If G is not a complete Graph, then the vertex connectivity number $k(G)$ is the minimum cardinality of a vertex cut. If G is complete graph K_p it is known that $k(G)=p-1$

Definition: Let $G=(V,E)$ be a graph. An equitable dominating set S of a graph G is called 2-equitable dominating set (2-ed-set) if for any vertex v in G either $v \in S$ or v is equitable dominated by at least 2 vertices in S. The minimum cardinality of a an 2-equitable dominating set of G is called the 2-equitable domination number of G and is denoted by $\gamma_{\times 2e}(G)$.

2. MAIN RESULTS

Definition: A Set $S \subseteq V$ is called the neighborhood Connected 2-equitable dominating set (nc2ed-set) of a graph G if for every $u \in V(G)$ either $u \in S$ or u is equitable dominated by at least 2 vertices in S and the induced subgraph $\langle N(S) \rangle$ is connected, The minimum Cardinality of nc2ed-set G is called the neighborhood Connected 2-equitable domination of G and is denoted by $\gamma_{2nce}(G)$.

Examples: γ_{2nce} value for well known graphs

1) $\gamma_{2nce}(K_p)=2$

2) $$\gamma_{2nce}(K_{r,s}) = \begin{cases} \begin{cases} 4 & \text{if } r \text{ and } s \neq 2,\ |r-s| \leq 1 \\ 3 & \text{If } r \text{ and } s = 2 \end{cases} \\ r+s & \text{if } |r-s| \geq 2 \end{cases}$$

3) $$\gamma_{2nce}(W_p) = \gamma_{2nce}(C_{p-1})+1 = \begin{cases} \left\lfloor \dfrac{2p+1}{3} \right\rfloor & \text{If } p \equiv 0 (\text{mod } 3) \\ \left\lfloor \dfrac{2p-2}{3} \right\rfloor + 1 & \text{otherwise} \end{cases}$$

In the following proposition we determine the relation between the $\gamma_{2nce}(G)$ and the other invariant domination parameters

Proposition 2.1: For any graph G. $\gamma(G) \leq \gamma_{nce}(G) \leq \gamma_{2nce}(G)$.

Proposition 2.2: For any graph G. $\gamma_{2e}(G) \le \gamma_{2nce}(G)$.

Theorem 2.3: For any non-trivial path P_P, $\gamma_{2nce}(P_p)=\left\lceil \frac{2p}{3} \right\rceil$

Proof: Let $P_p = (v_1, v_2, \ldots, v_P)$ and $p=3k+r$ where $0 \le r \le 2$

Let $S = \{v_i \in V: i=3j, 3j+1, 0 \le j \le k\}$

Let $S_1 = \begin{cases} S & \text{If } p \equiv 0,1 \pmod 3 \\ S \cup \{v_P\} & \text{If } p \equiv 2 \pmod 3 \end{cases}$

Clearly S_1 is a nc2ed-set of P_p and hence $\gamma_{2nce}(P_p)=\left\lceil \frac{2p}{3} \right\rceil$. Further if S is any γ_{2nce}-set of P_p, then $N_e(S)$ contains all the internal vertices of P_p and hence $|S| \ge \left\lceil \frac{2p}{3} \right\rceil$ Thus $\gamma_{2nce}(P_p)=\left\lceil \frac{2p}{3} \right\rceil$

Corollary 2.4: For any non-trivial path P_p, $\gamma_{2nce}(P_p) = \gamma_{nce}(P_p)$ if and only if p=3.

Proof: Since $\gamma_{nce}(P_p) = \left\lceil \frac{p}{3} \right\rceil$ the corollary follows.

Theorem 2.5: For the cycle C_p on p vertices,

$$\gamma_{2nce}(C_p) = \begin{cases} \left\lfloor \frac{2p}{3} \right\rfloor & \text{If } p \equiv 2 \pmod 3 \\ \left\lceil \frac{2p}{3} \right\rceil & \text{If } p \not\equiv 2 \pmod 3 \end{cases}$$

Proof: Let $C_p=(v_1, v_2, \ldots, v_p, v_1)$ and $p=2k+r$,Where $0 \le r \le 2$. Let $S=\{v_i : i=3j+1, 3j+2, 0 \le j \le k-1\}$

Let $S_1 = \begin{cases} S & \text{If } p \equiv 0 \pmod 3 \\ S \cup \{v_P\} & \text{If } p \equiv 1 \pmod 3 \\ S \cup \{v_{p-1}\} & \text{If } p \equiv 2 \pmod 3 \end{cases}$

Clearly S_1 is a nc2ed-set of C_p and hence

$$\gamma_{2nce}(C_p) = \begin{cases} \left\lfloor \frac{2p}{3} \right\rfloor & \text{If } p \equiv 2 \pmod 3 \\ \left\lceil \frac{2p}{3} \right\rceil & \text{If } p \not\equiv 3 \pmod 3 \end{cases}$$

Now let S be any γ_{2nce}-set of C_p. Then

$$N_e(S) = \begin{cases} P_{p-1} & \text{if } p \equiv 2 \pmod 3 \\ C_p & \text{if } p \not\equiv 2 \pmod 3 \end{cases}$$

Hence

$$|S| \geq \begin{cases} \left\lfloor \frac{2p}{3} \right\rfloor & \text{If } p \equiv 2 \pmod 3 \\ \left\lceil \frac{2p}{3} \right\rceil & \text{If } p \not\equiv 2 \pmod 3 \end{cases}$$

and the result follows.

Corollary 2.6: $\gamma_{2nce}(C_p) = \gamma_{nce}(C_p)$ iff and only if p=5.

Proof: Since

$$\gamma_{2nce}(C_p) = \begin{cases} \left\lceil \frac{p}{2} \right\rceil & \text{If } p \not\equiv 3 \pmod 4 \\ \left\lfloor \frac{p}{2} \right\rfloor & \text{If } p \equiv 3 \pmod 4 \end{cases}$$

The result follows.

Proposition 2.7: Any nce2d set contain all the pendent vertices of G.

Proof: Suppose the graph contains support vertices then by definition of nce2d-set, all the pendent vertices of G contains nce2d-set.

Proposition 2.8: $\gamma_{2nce}(G) \leq p$ the equality holds if and only if $G \cong K_2$

Proof: Suppose γ_{2nce}= p assume that $G \cong K_2$

Then G has at least three vertices u, v and w such that u and v are adjacent and w is not adjacent to one of u and v suppose w is not adjacent to u. This implies that V−{u} is a neighbourhood connected equitable 2-domination set of G, a contradiction. Hence G is isomorphic to K_2 converse is obvious.

Definition 2.9: A set S is minimal neighbourhood connected 2-equitable dominating set of G, if for any vertex u∈S, S−{u} is not a neighbourhood connected 2-equitable dominating set of graph G

Lemma 2.10: A super set of a nc2ed-set is a minimal nc2ed-set.

Proof: Let S be a nce2d-set of a graph G and Let S_1=S∪{v} where v∈V−S. Clearly v∈$N_e(S)$ and S_1 is a 2equitable dominating set of G. Now, let x,y ∈$N_e(S_1)$. If x, y∈$N_e(S)$ then any x-y path in $N_e(S)$ is a x-y path in $N_e(S_1)$. If x∈$N_e(S)$ and y∉$N_e(v)$ and x-v path in $N_e(S)$ followed by the edge v is a x-y path in $N_e(S_1)$. Also if x, y ∉$N_e(S)$ then (x, v, y) is a x-y path in $N_e(S_1)$. Thus ⟨$N_e(S_1)$⟩ is connected so that S_1 is a nce2d-set of G.

Theorem 2.11: A nc2ed-set S of a graph G is a minimal nc2ed-set if and only if for every u∈S one of the following holds.

1) $|N_e(u) \cap S| \leq |$.

2) There exists a vertex v∈ (V−S)∩$N_e(u)$ such that $|N_e(v) \cap S|=2$.

3) There exist two vertices x,y∈$N_e(S)$ such that every x-y path in ⟨$N_e(S)$⟩ contains at least one vertex of $N_e(S)$–$N_e(S-\{u\})$.

Proof: Let S be a minimal nce2d-set and let u∈S, let S_1=S−{u}. Then S_1 is not a nc2ed-set. This gives either S_1 is not a 2equitable dominating set or ⟨$N_e(S)$⟩ is disconnected. If S_1 is not an 2equitable dominating set then there exists a vertex v∈V−S_1 such that $|N_e(v) \cap S_1| \leq 1$. If v = u then $|N_e(u) \cap (S-\{u\}| \leq 1$ which gives $|N_e(u) \cap S| \leq 1$. Suppose u≠v. If $|N_e(v) \cap S_1|<1$ then $|N_e(v) \cap S|<1$ and hence S is not an 2equitable dominating set which is a contradiction. Hence $|N_e(v) \cap S|=1$. Thus v∈$N_e(u)$. So v∈(V−S)∩$N_e(u)$ such that $|N_e(v) \cap S|=2$. If ⟨$N_e(S_1)$⟩ is disconnected then there exist two vertices x, y$\in N_e(S_1)$ such that there is no x-y path in $\langle N_e(S_1)\rangle$ since $\langle N_e(S)\rangle$ is connected, it follows that every x-y in $\langle N_e(S_1)\rangle$ contains at least one vertex of $N_e(S)-N_e(S-\{u\})$. Conversely, if S is nc2ed-set of G satisfying the conditions of the theorem, then S is 1-minimal and hence result follows above lemma.

Theorem 2.12: Let G be a graph with P≥4 then $\gamma_{2nce}(G)\geq\left(\frac{2p+1-q}{2}\right)$ and this bound is sharp.

Proof: Let S be a γ_{2nce}-set of G. Then each vertex of V–S is equitable adjacent to at least two vertices in S. If G is not a star then since $\langle N_e(S)\rangle$ is connected either V–S or S contains at least one equitable edge. Hence the number of equitable edges $q\geq 2|V-S|+1=2p-2\gamma_{2nce}+1$ then $\gamma_{2nce}\geq\frac{2p+1-q}{2}$. The bound is sharp for C_5 and K_2.

Theorem 2.13: For any graph G, $\gamma_{2nce}(G)\geq\frac{2p}{(\Delta_e+2)}$

Proof: Let S be a minimum nc2ed-set and let k be the number of edges between S and V–S. Since the degree of each vertex in S is atmost Δ_e, $k\leq\Delta_e\gamma_{2nce}$. But since each vertex in V–S is adjacent to at least 2 vertices in S, $k\geq 2(p-\gamma_{2nce})$ combining these two inequalities produce

$$\gamma_{2nce}(G)\geq\frac{2p}{(\Delta_e+2)}.$$

Definition2.14: A colouring of a graph G is an assignment of colours to the vertices of G such that no two adjacent vertices receive the same colour, the minimum integer k for which a graph G is k-colourable is called the chromatic number of G and is denoted by $\chi(G)$.

Theorem 2.15: For any graph G, $\gamma_{2nce}(G)+\chi(G)\leq 2p$ and equality holds if and only if G is isomorphic to K_2.

Proof: The inequality is obvious, let $\gamma_{2nce}(G)+\chi(G)=2p$. This implies $\gamma_{2nce}(G)=p$ and $\chi(G)=p$. Hence G is isomorphic to K_2. The converse is obvious.

Theorem 2.16: Let G be a graph. Then $\gamma_{2nce}(G)+\chi(G)=2p-1$ if and only if G is isomorphic to K_3.

Proof: Let us assume that $\gamma_{2nce}(G)+\chi(G)=2p-1$. This is possible only if (i) $\gamma_{2nce}(G)=2p$ and $\chi(G)=2p-1$ or (ii) $\gamma_{2nce}(G)=p-1$ and $\chi(G)=p$. Since the condition (i) is impossible, condition (ii) holds. Thus implies G is a complete graph with $\gamma_{2nce}(G)=p-1$. Then p =3 and hence G is isomorphic to K_3. The converse is obvious.

Definition2.17: Let $H(v_1, v_2, \ldots, v_p)$ denotes the graph obtained from the graph H by pasting v_i edges to the vertex $v_i\in V(H)$, $1\leq i\leq p$.

Theorem 2.18: For any graph, $\gamma_{2nce}(G)+\chi(G)=2p-2$ if and only if G is isomorphic to K_4, or P_3 or $K_3(1, 0, 0)$.

Proof: Let us assume $\gamma_{2nce}(G)+\chi(G)=2p-2$. This is possible only if $\gamma_{2nce}(G)=p$ and $\chi(G)=p-2$ or $\gamma_{2nce}(G)=p-1$ and $\chi(G)=p-1$ or $\gamma_{2nce}(G)=p-2$ and $\chi(G)=p$. Let $\gamma_{2nce}(G)=p$ and $\chi(G)=p-2$. Since $\gamma_{2nce}(G)=p$ which gives G is isomorphic to K_2, and hence $\chi(G)=2\neq p-2$ which is a contradiction. Suppose $\gamma_{2nce}(G)=p-1$ and $\chi(G)=p-1$. Since $\chi(G)=p-1$, G contains a complete sub graph K on (p–1) vertices. Let $V(K)=\{v_1, v_2, \ldots\ldots\ldots v_{p-1}\}$ and $V(G)-V(K)=\{v_p\}$. Then v_p is equitable adjacent to v_i for some vertex $v_i\in V(K)$. If $deg_e(v_p)=1$ and $p\geq 4$ then $\{v_i, v_j, v_p\}$, $i\neq j$ is a γ_{2nce}-set of G. Hence p=4 and $K=K_3$. Thus G is isomorphic to $K_3(1, 0, 0)$. If $deg_e(v_p)=1$ and p = 3 then G is isomorphic to P_3. If $deg_e(v_p)>1$ then $\gamma_{2nce}=2$. Then p = 3 which gives G is isomorphic to K_3 which is a contradiction to $\chi(G)=p-1$.

Suppose $\gamma_{2nce}(G)=p-2$ and $\chi(G)=p$. since $\chi(G)=p$, isomorphic to K_p. But $\gamma_{2nce}(K_p)=2$. Therefore p=4. Hence G is isomorphic to K_4. The converse is obvious

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