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ON FOLDNESS OF INTUITIONISTIC FUZZY POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRAS

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ABSTRACT

In this paper, we derive the intuitionistic fuzzy n-fold BCK-ideal of X, intuitionistic fuzzy n-fold positive implicative ideal of BCK-algebra X and then discuss the related properties. We show that every intuitionistic fuzzy n-fold positive implicative ideal which is an intuitionistic fuzzy n-fold weak commutative ideal is an intuitionistic fuzzy n-fold implicative ideal. We gave characterizations of intuitionistic fuzzy n-fold Positive implicative ideals and establish the extension property for intuitionistic fuzzy n-fold Positive implicative ideals of BCK-algebras.

Key words: *BCK-algebra, Intuitionistic fuzzy ideal, Intuitionistic fuzzy positive implicative ideal.*

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1. INTRODUCTION:

For the general development of BCK-algebras, the ideal theory plays an important role. In 1999, Hung and Chen [5] introduced the notion of n-fold positive implicative ideals. The aim of this paper is to discuss the intuitionistic fuzzification of an n-fold BCK-ideal and n-fold positive implicative ideals of BCK-algebras. We define the notion of intuitionistic fuzzy n-fold BCK-ideals, intuitionistic fuzzy n-fold positive implicative ideals of BCK-algebras and then discuss the related properties. We show that every intuitionistic fuzzy n-fold positive implicative ideal which is an intuitionistic fuzzy n-fold weak commutative ideal is an intuitionistic fuzzy n-fold implicative ideal. Using level sets, we give characterizations of an intuitionistic fuzzy n-fold positive implicative ideal of BCK-algebras. Finally we establish the extension property for intuitionistic fuzzy n-fold positive implicative ideals in BCK-algebras.

2. PRELIMINARIES:

First we present the fundamental definitions. By a BCK-an algebra (see [8, 9]) we mean an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following axioms:

(BCK-1) $(x*y)*(x*z) \le (z*y)$,
(BCK-2) $x*(x*y) \le y$,
(BCK-3) $x \le x$,
(BCK-4) $x \le y, y \le x \Rightarrow x = y$,
(BCK-5) $0 \le x$, for every $x, y, z \in X$.

We can define a binary relation $\le$ on X by letting $x \le y$ if and only if $x*y=0$. Then $(X,\le)$ is a partially ordered set with least element 0. In any BCK-algebra X the following hold
(P1) $x*0 = x$,
(P2) $x*y \le x$,
(P3) $(x*y)*z = (x*z)*y$,
(P4) $(x*z)*(y*z) \le x*y$,
(P5) $x*(x*(x*y)) = x*y$,
(P6) $x \le y \Rightarrow x*z \le y*z$ and $z*y \le z*x$, for every $x, y, z \in X$.

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Throughout this paper X will always mean a BCK-algebra unless otherwise specified. A BCK-algebra X is said to be positive implicative if $(x * y) * z = (x * z) * (y * z)$ for every $x, y, z \in X$. A non-empty sub-set I of X is said to be sub-algebra of X if for $x, y \in X \Rightarrow x * y \in X$ A non-empty subset I of X is called an ideal of X if (I_1) $0 \in I$ (I_2) $x * y$ and $y \in I \Rightarrow x \in I$ for every $x, y \in X$

A non-empty sub-set I of X is said to be n-fold BCK- ideal if (I_1) and (I_3) there exists a fixed $n \in X$ such that $(x * y^{n+1}) * z \in I$ and $z \in I \Rightarrow x * y^n \in I$ for every $x, y, z \in X$. A non-empty sub-set I of X is said to be an n-fold positive implicative ideal if (I_1) and there exists a fixed $n \in N$ such that

(I_4) $(x * y) * z^n \in I$ and $y * z^n \in I \Rightarrow x * z^n \in I$ for every $x, y, z \in X$.

For any elements x and y of X, $x * y^n$ denotes $(-------((x * y) * y) * ------) * y$ in which 'y' occurs n-times.

Let μ and λ be the fuzzy sets of X. For $s, t \in [0,1]$ the set U (μ; s) = { $x \in X$ / μ(x) ≥ s} is called upper s- level cut of μ and the set L (λ; t) = { $x \in X$ / λ(x) ≤ t} is called lower t-level cut of λ and can used to the characterizations of μ and λ.

As an important generalization of the notion of fuzzy sets in X, Atanassov [1, 2] introduced the concept of an intuitionistic fuzzy set (IFS for short) defined by "An intuitionist fuzzy set A in a non- empty set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x)) / x \in X\}$, where the function $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denoted the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively and $0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$. For the sake of simplicity, we use the symbol form $A = (X, \mu_A, \lambda_A)$ (or) $A = (\mu_A, \lambda_A)$".

Definition: 2.1 [12] An IFS $A = (X, \mu_A, \lambda_A)$ in X is an intuitionistic fuzzy ideal (IF-ideal) of X, if it satisfies

(IF-1) $\mu_A(0) \ge \mu_A(x)$ and $\lambda_A(0) \le \lambda_A(y)$
(IF-2) $\mu_A(x) \ge \min\{\mu_A(x*y), \mu_A(x)\}$
(IF-3) $\lambda_A(x) \le \max\{\lambda_A(x*y), \lambda_A(y)\}$ for all $x, y \in X$

Definition: 2.2 [12] An IFS $A = (X, \mu_A, \lambda_A)$ in X is an intuitionistic fuzzy sub- algebra of X, if it satisfies

(i) $\mu_A(x*y) \ge \min\{\mu_A(x), \mu_A(y)\}$
(ii) $\lambda_A(x*y) \le \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in X$.

Theorem: 2.3. [12] Let $A = (X, \mu_A, \lambda_A)$ intuitionistic fuzzy ideal of X, if $x \le y$ in X, then $\mu_A(x) \ge \mu_A(y)$ and $\lambda_A(x) \le \lambda_A(y)$, that is, μ_A is an order-reversing and λ_A is an order- preserving.

Theorem 2.4 [12] An IFS $A = (X, \mu_A, \lambda_A)$ in X is an intuitionistic fuzzy ideal of X, if for $x, y, z \in X$, $x * y \le z \Rightarrow \mu_A(x) \ge \min\{\mu_A(y), \mu_A(z)\}$ and $\lambda_A(x) \le \max\{\lambda_A(y), \lambda_A(z)\}$.

Definition: 2.5 [13] An IFS $A = (X, \mu_A, \lambda_A)$ in X is an intuitionistic fuzzy n-fold implicative

$(IFI^n - \text{ideal})$ ideal of X, if it satisfies

$(IFI^n 1)$ $\mu_A(0) \ge \mu_A(x)$ and $\lambda_A(0) \le \lambda_A(x)$ and there exists a fixed $n \in N$ such that

$(IFI^n 2)$ $\mu_A(x) \geq \min\{\mu_A((x*(y*x^n))*z), \mu_A(z)\}$

$(IFI^n 3)$ $\lambda_A(x) \leq \max\{\lambda_A((x*(y*x^n))*z), \lambda_A(z)\}$, for every $x, y, z \in X$.

Definition: 2.6 [13] An IFS $A=(X,\mu_A,\lambda_A)$ in X is an intuitionistic fuzzy n-fold commutative ideal $(IFCI^n-\text{ideal})$ of X if it satisfies

$(IFCI^n 1)$ $\mu_A(0) \geq \mu_A(x)$ $\lambda_A(0) \leq \lambda_A(x)$ and there exist a fixed $n \in N$ such that

$(IFCI^n 2)$ $\mu_A(x*(y*(y*x^n))) \geq \min\{\mu_A((x*y)*z), \mu_A(z)\}$

$(IFCI^n 3)$ $\lambda_A(x*(y*(y*x^n))) \leq \max\{\lambda_A((x*y)*z), \lambda_A(z)\}$ for all $x, y, z \in X$.

Definition: 2.7 [13] An IFS $A=(X,\mu_A,\lambda_A)$ in X is an intuitionistic fuzzy n-fold weak commutative ideal $(IFWC^n-\text{ideal})$ of X if it satisfies

$(IFWCI^n\text{-}1)$ $\mu_A(0) \geq \mu_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$ and there exists a fixed $n \in N$ such that

$(IFWCI^n\text{-}2)$ $\mu_A(y*(y*x)) \geq \min\{\mu_A((x*(x*y^n)*z), \mu_A(z)\}$

$(IFWCI^n\text{-}3)$ $\lambda_A(y*(y*x)) \leq \max\{\lambda_A((x*(x*y^n)*z), \lambda_A(z)\}$ for all $x, y, z \in X$.

3. INTUITIONISTIC FUZZY n-FOLD BCK-IDEAL OF BCK-ALGEBRAS:

Definition: 3.1. An IFS $A=(X,\mu_A,\lambda_A)$ in X is an intuitionistic fuzzy n-fold BCK-ideal of X if it satisfies

$(BCKI^n 1)$ $\mu_A(0) \geq \mu_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$ and there exists a fixed $n \in N$ such that

$(BCKI^n 2)$ $\mu_A(x*y^n) \geq \min\{\mu_A((x*y^{n+1})*z), \mu_A(z)\}$

$(BCKI^n 3)$ $\lambda_A(x*y^n) \leq \max\{\lambda_A((x*y^{n+1})*z), \lambda_A(z)\}$ for all $x, y, z \in X$.

Proposition: 3.2. Every intuitionistic fuzzy n-fold BCK- ideal of X is an intuitionistic fuzzy deal of X.

Proof: Let $A=(X,\mu_A,\lambda_A)$ be an intuitionist fuzzy n-fold BCK- ideal of X

Put $y = 0$ in $(BCKI^n 2)$ and $(BCKI^n 3)$ we get

$$\mu_A(x)=\mu_A(x*0^n) \geq \min\{\mu_A((x*0^{n+1})*z), \mu_A(z)\} \geq \min\{\mu_A((x*z), \mu_A(z)\}$$

and

$$\lambda_A(x)=\lambda_A(x*0^n) \geq \max\{\lambda_A((x*0^{n+1})*z), \lambda_A(z)\} \geq \max\{\lambda_A((x*z), \lambda_A(z)\}.$$

Thus $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy ideal of X.

The following example shows that the converse of theorem 3.2 may not be true.

Example: 3.3. Let $X = N \cup \{0\}$, where N is the set of natural numbers , in which the operation $*$ is defined by $x * y = \max\{0,\ x - y\}$ for all $x, y \in X$ .Then X is a BCK-algebra [5, Example 1.3]

Let $A=(X,\mu_A,\lambda_A)$ be an IFS in X given by $\mu_A(0)=0.8>0.3=\mu_A(x)$ and $\lambda_A(0)=0.1<0.4=\lambda_A(x)$,

for all $x(\neq 0) \in X$ .Then A is an IF-ideal of X but $A=(X,\mu_A,\lambda_A)$ is not an intuitionistic fuzzy 2-fold BCK-ideal of X, because

$\mu_A(5*2^2)=\mu_A(1)=0.3<0.8=\mu_A(0)=\min\{\mu_A((5*2^3))*0),\mu_A(0)\}$

and

$\lambda_A(5*2^2)=\lambda_A(1)=0.4>0.1=\lambda_A(0)=\max\{\lambda_A((5*2^3))*0),\lambda_A(0)\}$,

we give a condition for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy n-fold BCK-ideal.

Proposition: 3.4. Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy ideal of X.Then A is an intuitionistic fuzzy n-fold BCK-ideal of X if and only if it satisfies the following inequalities

$\mu_A(x*y^n)\geq\mu_A(x*y^{n+1})$ and $\lambda_A(x*y^n)\leq\lambda_A(x*y^{n+1})$ for all $x, y\in X$.

Proof: Suppose that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold BCK-ideal of X

Put $z=0$ in $(BCKI^n 2)$ and $(BCKI^n 3)$ we get

$\mu_A(x*y^n)\geq\min\{\mu_A((x*y^{n+1})*0),\mu_A(0\}=\min\{\mu_A(x*y^{n+1}),\mu_A(0)\}=\mu_A(x*y^{n+1}\}$ and

$\lambda_A(x*y^n)\leq\max\{\lambda_A((x*y^{n+1})*0),\lambda_A(0\}=\max\{\lambda_A(x*y^{n+1}),\lambda_A(0)\}=\lambda_A(x*y^{n+1}\}$

Therefore $\mu_A(x*y^n)\geq\mu_A(x*y^{n+1})$ and $\lambda_A(x*y^n)\leq\lambda_A(x*y^{n+1})$ for all $x, y\in X$.

Conversely, suppose that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy ideal of X satisfies the inequalities

$\mu_A(x*y^n)\geq\mu_A(x*y^{n+1})$ and $\lambda_A(x*y^n)\leq\lambda_A(x*y^{n+1})$ for all $x, y\in X$.

Using (IF2) and (IF3) we get

$\mu_A(x*y^n)\geq\mu_A(x*y^{n+1})\geq\min\{\mu_A((x*y^{n+1})*z),\mu_A(z)\}$ and

$\lambda_A(x*y^n)\leq\lambda_A(x*y^{n+1})\leq\max\{\lambda_A((x*y^{n+1})*z),\lambda_A(z)\}$ for all $x, y, z\in X$.

Thus $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold BCK-ideal of X

Corollary: 3.5. Every intuitionistic fuzzy n-fold BCK-ideal $A=(X,\mu_A,\lambda_A)$ of X satisfies the inequalities $\mu_A(x*y^n)\geq\mu_A(x*y^{n+k})$ and $\lambda_A(x*y^n)\leq\lambda_A(x*y^{n+k})$ for all $x, y\in X$. and $k\in N$.

Proof: Using the proposition 3.4, the proof is straightforward by induction

4. INTUITIONISTIC FUZZY n-FOLD POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRAS:

Definition: 4.1. An IFS $A=(X,\mu_A,\lambda_A)$ in X is an intuitionistic fuzzy n-fold positive implicative ideal

($IFPI^n$ - ideal) of X if it satisfies

$(IFPI^n 1)$ $\mu_A(0)\geq\mu_A(x)$, $\lambda_A(0)\leq\lambda_A(x)$ and there exists a fixed $n\in N$ such that

$(IFPI^n 2)$ $\mu_A(x*z^n)\geq\min\{\mu_A((x*y)*z^n),\ \mu_A(y*z^n)\}$

$(IFPI^n 3)$ $\lambda_A(x*z^n)\leq\max\{\lambda_A((x*y)*z^n),\ \lambda_A(y*z^n)\}$ for all $x, y, z\in X$.

An intuitionistic fuzzy 1-fold positive implicative ideal is an intuitionistic fuzzy positive implicative ideal

Example: 4.2 Let $X=\{0,1,2\}$ be a BCK-algebra with the following cayley table

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Define an IFS $A=(X,\mu_A,\lambda_A)$ in X by $\mu_A(0)=0.8,\mu_A(1)=0.7,\mu_A(2)=0.3$ and $\lambda_A(0)=0.1,\lambda_A(1)=0.2,\mu_A(2)=0.4$. Then $A=(X,\mu_A,\lambda_A)$ is an intuitionist fuzzy n-fold positive implicative ideal of X for all $n\in N$

Theorem: 4.3 Every intuitionistic fuzzy n-fold positive implicative ideal of X must be tuitionistic fuzzy ideal of X.

Proof: Let $A=(X,\mu_A,\lambda_A)$ be an intuitionist fuzzy n-fold Positive implicative ideal of X. Put $z=0$ in $(\text{IFPI}^n 2)$ and $(\text{IFPI}^n 3)$ we get

$$\mu_A(x)=\mu_A(x*0^n)\geq\min\{\mu_A((x*y)*0^n),\mu_A(y*0^n)\}=\min\{\mu_A(x*y),\mu_A(y)\}$$

and

$$\lambda_A(x)=\lambda_A(x*0^n)\geq\max\{\lambda_A((x*y)*0^n),\lambda_A(y*0^n)\}=\max\{\lambda_A(x*y),\lambda_A(y)\}$$

Thus $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy ideal of X.

The following example shows that the converse of the Theorem 4.3 may not be true.

Example: 4.4 Let $X=N\cup\{0\}$, where N is the set of natural numbers , in which the operation $*$ is defined by $x*y=\max\{0,\ x-y\}$ for all $x,y\in X$. Then X is a BCK-algebra ([5], 1.3).

Let $A=(X,\mu_A,\lambda_A)$ be an IFS in X given by $\mu_A(0)=0.4>0.1=\mu_A(x)$ and $\lambda_A(0)=0.1<0.4=\lambda_A(x)$, for all $x(\neq 0)\in X$. Then A is an IF-ideal of X, but $A=(X,\mu_A,\lambda_A)$ is not an intuitionistic fuzzy 2-fold positive implicative-ideal of X, because

$$\mu_A(13*5^2)=\mu_A(3)=0.3<0.8=\mu_A(0)=\min\{\mu_A((13*3)*5^2)),\ \mu_A(3*5^2)\}$$

and

$$\lambda_A(13*5^2)=\lambda_A(3)=0.3<0.8=\lambda_A(0)=\max\{\lambda_A((13*3)*5^2)),\ \lambda_A(3*5^2)\}$$

Corollary: 4.5. Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy n-fold positive implicative ideal of if $x\leq y$ in X, then $\mu_A(x)\geq\mu_A(y)$ and $\lambda_A(x)\leq\lambda_A(y)$, that is, μ_A is an order-reversing and λ_A is an order-preserving.

We give some conditions for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy n-fold positive implicative ideal.

Theorem: 4.6 Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy ideal of X.Then A is an intuitionistic fuzzy n-fold positive implicative ideal of X if and only if it satisfies the inequalities

$$\mu_A((x*z^n)*(y*z^n))\geq\mu_A((x*y)*z^n)$$

and

$$\lambda_A((x*z^n)*(y*z^n))\leq\lambda_A((x*y)*z^n)$$ for all $x,y,z\in X$.

Proof: Assume that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X. By Theorem 4.3, that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy ideal of X. Let $x, y, z \in X$ and $a=x*(y*z^n)$ and $b = x*y$.

Since $((x*(y*z^n))*(x*y))*z^n) \le (y*(y*z^n))*z^n$. By Corollary 4.5, we have

$$\mu_A(((x*(y*z^n))*(x*y))*z^n)) \ge \mu_A((y*(y*z^n))*z^n)$$

and

$$\lambda_A(((x*(y*z^n))*(x*y))*z^n)) \le \lambda_A((y*(y*z^n))*z^n)$$

Then $\mu_A((a*b)*z^n) = \mu_A(((x*(y*z^n))*(x*y))*z^n)$

$$\ge \mu_A((y*(y*z^n))*z^n) \quad \text{(by BCK-1)}$$
$$= \mu_A((y*z^n)*(y*z^n)) \quad \text{(by P3)}$$
$$= \mu_A(0). \quad \text{(by BCK-3)}$$

and so $\mu_A((a*b)*z^n) = \mu_A(0)$.

And $\lambda_A((a*b)*z^n) = \lambda_A(((x*(y*z^n))*(x*y))*z^n)$

$$\le \lambda_A((y*(y*z^n))*z^n)) \quad \text{(by BCK-1)}$$
$$= \lambda_A((y*z^n)*(y*z^n)) \quad \text{(by P3)}$$
$$= \lambda_A(0). \quad \text{(by BCK-3)}$$

and so $\lambda_A((a*b)*z^n) = \lambda_A(0)$.

Using (P3), $(\mathrm{IFPI}^n 2)$ and $(\mathrm{IFPI}^n 3)$ we obtain

$$\begin{aligned}
\mu_A((x*z^n)*(y*z^n)) &= \mu_A((x*(y*z^n))*z^n) = \mu_A(a*z^n) \\
&\ge \min\{\mu_A((a*b)*z^n), \mu_A(b*z^n)\} \\
&= \min\{\mu_A(0), \mu_A(b*z^n)\} \\
&= \mu_A(b*z^n) = \mu_A((x*y)*z^n) \text{ and}
\end{aligned}$$

$$\begin{aligned}
\lambda_A((x*z^n)*(y*z^n)) &= \lambda_A((x*(y*z^n))*z^n) = \lambda_A(a*z^n) \\
&\le \max\{\lambda_A((a*b)*z^n), \lambda_A(b*z^n)\} \\
&= \max\{\lambda_A(0), \lambda_A(b*z^n)\} \\
&= \lambda_A(b*z^n) = \lambda_A((x*y)*z^n)
\end{aligned}$$

Thus $\mu_A((x*z^n)*(y*z^n)) \ge \mu_A((x*y)*z^n)$ and $\lambda_A((x*z^n)*(y*z^n)) \le \lambda_A((x*y)*z^n)$ for all $x \in X$.

Conversely, suppose that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy ideal of X satisfies the inequalities

$\mu_A((x*z^n)*(y*z^n)) \geq \mu_A((x*y)*z^n)$

and

$\lambda_A((x*z^n)*(y*z^n)) \leq \lambda_A((x*y)*z^n)$ for all $x,y,z \in X$.

Using (IF-2) and (IF-3) we obtain

$\mu_A(x*z^n) \geq \min\{\mu_A((x*z^n)*(y*z^n)), \mu_A(y*z^n)\} \geq \min\{\mu_A((x*y)*z^n), \mu_A(y*z^n)\}$

and

$\lambda_A(x*z^n) \leq \max\{\lambda_A((x*z^n)*(y*z^n)), \lambda_A(y*z^n)\} \leq \max\{\lambda_A((x*y)*z^n), \lambda_A(y*z^n)\}$

for all $x, y, z \in X$. Thus $A=(X,\mu_A,\lambda_A)$ is an intuitionist fuzzy n-fold positive implicative ideal of X.

Proposition: 4.7 Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy ideal of X. Then A is an Intuitionistic fuzzy n-fold positive implicative ideal of X if and only if it satisfies the inequalities

$\mu_A(x*y^n) \geq \mu_A((x*y)*y^n)$ and $\lambda_A(x*y^n) \leq \lambda_A((x*y)*y^n)$ for all $x, y, z \in X$.

Proof: Assume that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold Positive implicative ideal of X. Put $z = y$ in $(\mathrm{IFPI}^n 2)$ and $(\mathrm{IFPI}^n 3)$ we get

$\mu_A(x*y^n) \geq \min\{\mu_A((x*y)*y^n), \mu_A(y*y^n)\} = \min\{\mu_A((x*y)*y^n)), \mu_A(0)\} = \mu_A((x*y)*y^n)$

$\lambda_A(x*y^n) \leq \max\{\lambda_A((x*y)*y^n), \lambda_A(y*y^n)\} = \max\{\lambda_A((x*y)*y^n)), \lambda_A(0)\} = \lambda_A((x*y)*y^n)$.

Thus $\mu_A(x*y^n) \geq \mu_A((x*y)*y^n)$ and $\lambda_A(x*y^n) \leq \lambda_A((x*y)*y^n)$, for all $x, y \in X$

Conversely suppose that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy ideal of X satisfies the inequalities $\mu_A(x*y^n) \geq \mu_A((x*y)*y^n)$ and $\lambda_A(x*y^n) \leq \lambda_A((x*y)*y^n)$ for all $x, y \in X$.

Since $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$

Now we can prove $\mu_A(x*z^n) \geq \min\{\mu_A((x*y)*z^n), \mu_A(y*z^n)\}$

$\lambda_A(x*z^n) \leq \max\{\lambda_A((x*y)*z^n), \lambda_A(y*z^n)\}$, for all $x, y, z \in X$.

In contrary, there exist $x_0, y_0 \in X$ such that

$\mu_A(x_0*y_0{}^n) < \min\{\mu_A((x_0*y_0)*y_0{}^n), \mu_A(y_0*y_0{}^n)\}$

$\Rightarrow \mu_A(x_0*y_0{}^n) < \mu_A((x_0*y_0)*y_0{}^n)$, which is a contradiction.

Therefore $\mu_A(x*z^n) \geq \min\{\mu_A((x*y)*z^n), \mu_A(y*z^n)\}$, for all $x, y, z \in X$.

Similarly we can prove $\lambda_A(x*z^n) \leq \min\{\lambda_A((x*y)*z^n), \lambda_A(y*z^n)\}$ for all $x, y, z \in X$.

Thus A is an intuitionistic fuzzy n-fold positive implicative ideal of X.

Proposition: 4.8 Let $A=(X,\mu_A,\lambda_A)$ be an intuitionist fuzzy set of X. Then A is an intuitionistic fuzzy n-fold positive implicative ideal of X if and only if it is an intuitionistic fuzzy n-fold BCK-ideal of X.

Proof: Assume $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X. By Corollary 4.3, that $A=(X,\mu_A,\lambda_A)$ is an intuitionist fuzzy ideal of X. Putting $z = y$ in $(IFPI^n 2)$ and $(IFPI^n 3)$ we get

$$\mu_A(x*y^n)\geq\min\{\mu_A((x*y)*y^n),\mu_A(y*y^n)=\min\{\mu_A(x*y^{n+1}),\mu_A(0)\}\}=\mu_A(x*y^{n+1})$$
$$\lambda_A(x*y^n)\leq\max\{\lambda_A((x*y)*y^n),\lambda_A(y*y^n)=\max\{\mu_A(x*y^{n+1}),\lambda_A(0)\}\}=\lambda_A(x*y^{n+1})$$

By proposition 3.4, A is an intuitionistic fuzzy n-fold BCK-ideal of X. Conversely, suppose that A is an intuitionistic fuzzy n-fold BCK-ideal of X. By proposition 3.2, that $A=(X,\mu_A,\lambda_A)$ is an intuitionist fuzzy ideal of X, by Theorem 2.3 μ_A is order reversing and λ_A is order preserving. It follows from (P3) and (P4) that

$$\mu_A((x*z^{2n})*(y*z^n))=\mu_A(((x*z^n)*z^n)*(y*z^n))=\mu_A(((x*z^n)*(y*z^n)*z^n)\geq\mu_A((x*y)*z^n).$$
$$\lambda_A((x*z^{2n})*(y*z^n))=\lambda_A(((x*z^n)*z^n)*(y*z^n))=\lambda_A(((x*z^n)*(y*z^n)*z^n)\leq\lambda_A((x*y)*z^n).$$

Using corollary 3.5, (IF2) and (IF3) we get

$$\begin{aligned}\mu_A(x*z^n)\geq\mu_A(x*z^{2n})&\geq\min\{\mu_A((x*z^{2n})*(y*z^n)),\mu_A(y*z^n)\}\\&\geq\min\{\mu_A((x*y)*z^n),\mu_A(y*z^n)\}\\\lambda_A(x*z^n)\leq\lambda_A(x*z^{2n})&\leq\max\{\lambda_A((x*z^{2n})*(y*z^n)),\lambda_A(y*z^n)\}\\&\leq\max\{\lambda_A((x*y)*z^n),\lambda_A(y*z^n)\}.\end{aligned}$$

Thus A is an intuitionistic fuzzy n-fold positive implicative ideal of X.

Theorem: 4.9 An IFS $A=(X,\mu_A,\lambda_A)$ in X is an intuitionistic fuzzy n-fold positive implicative ideal of X if and only if the non-empty upper s-level cut $U(\mu_A;s)$ and the non-empty lower t-level cut $L(\lambda_A;t)$ are n-fold positive implicative deals of X for any $s,t\in[0,1]$.

Proof: Assume $A=(X,\mu_A,\lambda_A)$ is intuitionist fuzzy n-fold positive implicative ideal of a X. For any $s,t\in[0,1]$, define the sets

$U(\mu_A;s)=\{x\in X/\mu_A(x)\geq s\}$ and $L(\lambda_A;t)=\{x\in X/\lambda_A(x)\leq t\}$.

Since $U(\mu_A;s)\neq\varphi$. Let $x\in U(\mu_A;s)\Rightarrow\mu_A(x)\geq s$. By definition we have $\mu_A(0)\geq\mu_A(x)$ for all $x\in X$ implies $0\in U(\mu_A;s)$. Let $x,y,z\in X$ be such that $((x*y)*z^n)\in U(\mu_A;s), y*z^n\in U(\mu_A;s)$
$\Rightarrow\mu_A((x*y)*z^n))\geq s$ and $\mu_A(y*z^n)\geq s$.

Since
$\mu_A(x*z^n)\geq\min\{\mu_A((x*y)*z^n),\mu_A(y*z^n)\}\geq\min\{s,s\}=s\Rightarrow\mu_A(x*z^n)\geq s\Rightarrow x*z^n\in U(\mu_A;s)$.
Let $x\in L(\lambda_A;t)\Rightarrow\lambda_A(x)\leq t$ since $\lambda_A(0)\leq\lambda_A(x)$ for all $x\in X$ imply $0\in L(\lambda_A;t)$. Further more if $(x*y)*z^n\in L(\lambda_A,t), y*z^n\in L(\lambda_A,t)$ then $\lambda_A((x*y)*z^n)\leq t$ and $\lambda_A(y*z^n)\leq t$.

Since $\lambda_A(x*z^n)\le\max\{\lambda_A((x*y)*z^n),\lambda_A(y*z^n)\}=\max\{t,t\}=t$, for all $x,y,z\in X$.

Therefore $\lambda_A(x*z^n)\le t\Rightarrow x*z^n\in L(\lambda_A;t)$. Thus $U(\mu_A;s)$ and $L(\lambda_A;t)$ are n-fold positive implicative ideals of X, for all $s,t\in[0,1]$.

Conversely suppose that $U(\mu_A;s)$ and $L(\lambda_A;t)$ are n-fold positive implicative ideals of X, for all $s,t\in[0,1]$. Put $\mu_A(x)=s,\lambda_A(x)=t$ for any $x\in X$.

Since $0\in U(\mu_A;s)\Rightarrow\mu_A(0)\ge s=\mu_A(x)$ and $0\in L(\lambda_A,t)\Rightarrow\lambda_A(0)\le t=\lambda_A(x)$ for all $x\in X$

Now we prove that $(IFPI^n 2)$ and $(IFPI^n 3)$. In contrary, there exists $x_0,y_0,z_0\in X$ such that

$\mu_A(x_0*z_0^n)<\min\{\mu_A((x_0*y_0)*z_0^n)),\mu_A(y_0*z_0^n)\}$.

Taking $s_0=\frac{1}{2}\left[\mu_A(x_0*z_0^n))+\min\{\mu_A((x_0*y_0)*z_0^n),\mu_A(y_0*z_0^n)\}\right]$,

It follows that $((x_0*y_0)*z_0^n),y_0*z_0^n\in U(\mu_A,s_0)$ but $x_0*z_0^n\notin U(\mu_A;s_0)$. This is a contradiction.

Similarly, we can prove $(IFPI^n 3)$. Thus $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X.

Theorem: 4.10 Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy ideal of X, then the following Conditions are equivalent:

(i) $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X.
(ii) $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold BCK-ideal of X.
(iii) $\mu_A(x*y^n)\ge\mu_A(x*y^{n+1})$ and $\lambda_A(x*y^n)\le\lambda_A(x*y^{n+1})$ for all $x,y\in X$.
(iv) $\mu_A((x*z^n)*(y*z^n))\ge\mu_A((x*y)*z^n)$ and $\lambda_A((x*z^n)*(y*z^n))\le\lambda_A((x*y)*z^n)$ for all $x,y,z\in X$.
(v) $U(\mu_A;s)$ and $L(\lambda_A;t)$ are n-fold positive implicative ideals of X for all $s,t\in[0,1]$.

Theorem: 4.11 If $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X then

(i) for all $x,y,a,b\in X,((x*y)*y^n)*a\le b\Rightarrow\mu_A(x*y^n)\ge\min\{\mu_A(a),\mu_A(b)\}$

and $\lambda_A(x*y^n)\le\max\{\lambda_A(a),\lambda_A(b)\}$

(ii) for all $x,y,z,a,b\in X$, $((x*y)*z^n)*a\le b\Rightarrow\mu_A((x*z^n)*(y*z^n))\ge\min\{\mu_A(a),\mu_A(b)\}$

and $\lambda_A((x*z^n)*(y*z^n))\le\max\{\lambda_A(a),\lambda_A(b)\}$

Proof: Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy n-fold positive implicative ideal of X. Let $x,y,z\in X$ be such that $((x*y)*y^n)*a\le b$ .Using theorem 2.4, we have

$\mu_A((x*y)*y^n)\ge\min\{\mu_A(a),\mu_A(b)\}$ and $\lambda_A((x*y)*y^n)\le\max\{\lambda_A(a),\lambda_A(b)\}$.

Put $z=y$ in $(IFPI^n 2)$ and $(IFPI^n 3)$ we get

$$\mu_A(x*y^n) \geq \min\{\mu_A((x*y)*y^n), \mu_A(y*y^n)\} = \min\{\mu_A((x*y)*y^n), \mu_A(0)\} = \mu_A((x*y)*y^n)$$
$$\geq \min\{\mu_A(a), \mu_A(b)\}$$

and

$$\lambda_A(x*y^n) \leq \max\{\lambda_A((x*y)*y^n), \lambda_A(y*y^n)\} = \max\{\lambda_A((x*y)*y^n), \lambda_A(0)\} = \lambda_A((x*y)*y^n)$$
$$\leq \max\{\lambda_A(a), \lambda_A(b)\}$$

(ii) Let $x, y, z \in X$ be such that $((x*y)*z^n)*a \leq b$. Since $A = (X, \mu_A, \lambda_A)$ intuitionist fuzzy n-fold positive implicative ideal of X, it follows from theorems 2.4 and 4.6. We obtain

$$\mu_A((x*z^n)*(y*z^n)) \geq \mu_A((x*y)*z^n) \geq \min\{\mu_A(a), \mu_A(b)\}$$

and

$$\lambda_A((x*z^n)*(y*z^n)) \leq \lambda_A((x*y)*z^n) \leq \max\{\lambda_A(a), \lambda_A(b)\}$$

This completes the proof.

Theorem: 4.12 Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy n-fold positive implicative -ideal of X, then so is $\neg A$, where $\neg A=(X,\mu_A,\bar{\mu}_A)$.

Theorem: 4.13 Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy n-fold positive implicative ideal of X then so is $\Diamond A$, where $\Diamond A=(X,\bar{\lambda}_A,\lambda_A)$.

Theorem: 4.14 $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X if and only if $\neg A=(X,\mu_A,\bar{\mu}_A)$ and $\Diamond A=(X,\bar{\lambda}_A,\lambda_A)$ are intuitionistic fuzzy n-fold positive implicative ideals of X.

Theorem: 4.15 Let $A=(X,\mu_A,\lambda_A)$ be an intuitionistic fuzzy set of X. If A is an intuitionistic fuzzy n-fold positive implicative ideal of X, then the sets $J=\{x\in X/\mu_A(x)=\mu_A(0)\}$ and $K=\{x\in X/\lambda_A(x)=\lambda_A(0)\}$ are n-fold positive implicative ideals of X.

Theorem: 4.16 An IFS $A=(X,\mu_A,\lambda_A)$ is both an intuitionistic fuzzy n-fold positive implicative ideal and an intuitionistic fuzzy n-fold weak commutative ideal of X then it is an intuitionistic fuzzy n-fold implicative ideal of X.

Proof: Let $x, y \in X$. Using ([13], 4.5(ii)), Corollary 4.5, (P3) and (BCK-3),

we have $\mu_A(x*(x*(y*x^n))) \geq \mu_A((y*x^n)*((y*x^n)*x^n)) \geq \mu_A((y*(y*x^n))*x^n))$

$$=\mu_A((y*x^n)*(y*x^n))=\mu_A(0)$$

and $\lambda_A(x*(x*(y*x^n))) \leq \lambda_A((y*x^n)*((y*x^n)*x^n)) \leq \lambda_A((y*(y*x^n))*x^n))$

$$= \lambda_A((y*x^n)*(y*x^n)) = \lambda_A(0).$$

Since $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy ideal of X from this we have

$$\mu_A(x) \geq \min\{\mu_A(x*(x*(y*x^n))), \mu_A(x*(y*x^n))\}$$
$$\geq \min\{\mu_A(0), \mu_A(x*(y*x^n)) = \mu_A(x*(y*x^n)) \text{ and}$$
$$\lambda_A(x) \leq \max\{\lambda_A(x*(x*(y*x^n))), \lambda_A(x*(y*x^n))\}$$

$$\leq \max\{\lambda_A(0), \lambda_A(x*(y*x^n))$$
$$= \lambda_A(x*(y*x^n))$$

So from ([13, 3.4)], $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold implicative ideal of X.

Theorem: 4.17 (Extension property for intuitionistic fuzzy n-fold positive implicative ideals)
Let $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ be an intuitionistic fuzzy ideals of X such that $A(0)=B(0)$ and $A \subseteq B$, that is, $\mu_A(0)=\mu_B(0), \lambda_A(0)=\lambda_B(0)$ and $\mu_A(x) \leq \mu_B(x), \lambda_A(x) \geq \lambda_B(x)$, for all $x \in X$. If $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X then so is B.

Proof: Suppose $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X. Using Proposition 3.4, it is sufficient to show that $B=(X,\mu_B,\lambda_B)$ satieties the inequalities

$\mu_B(x*y^n) \geq \mu_B(x*y^{n+1})$ and $\lambda_B(x*y^n) \leq \lambda_B(x*y^{n+1})$ for all $x, y \in X$.

Let $x, y \in X$ .Using (BCK-3), (P3) and Proposition 3.4, we get

$$\mu_B(0)=\mu_A(0)=\mu_A((x*(x*y^{n+1}))*y^{n+1}) \leq \mu_A((x*(x*y^{n+1}))*y^n)$$
$$=\mu_A((x*y^n)*(x*y^{n+1})) \leq \mu_B((x*y^n)*(x*y^{n+1})$$ and
$$\lambda_B(0)=\lambda_A(0)=\lambda_A((x*(x*y^{n+1}))*y^{n+1}) \geq \lambda_A((x*(x*y^{n+1}))*y^n)$$
$$=\lambda_A((x*y^n)*(x*y^{n+1})) \geq \lambda_B((x*y^n)*(x*y^{n+1}).$$

Since $B=(X,\mu_B,\lambda_B)$ is an intuitionistic fuzzy ideal of X, it follows from (IF1), (IF2) and (IF3) that

$$\mu_B(x*y^n) \geq \min\{\mu_B((x*y^n)*(x*y^{n+1})), \mu_B(x*y^{n+1})\}$$
$$\geq \min\{\mu_B(0), \mu_B(x*y^{n+1})\} = \mu_B(x*y^{n+1})$$

And $\lambda_B(x*y^n) \leq \max\{\lambda_B((x*y^n)*(x*y^{n+1})), \lambda_B(x*y^{n+1})\}$

$$\leq \max\{\lambda_B(0), \lambda_B(x*y^{n+1})\} = \lambda_B(x*y^{n+1}).$$

Thus $B=(X,\mu_B,\lambda_B)$ is an intuitionistic fuzzy n-fold positive implicative ideal of X.

REFERENCES:

[1] Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1)(1983), 87-96.

[2] Atanassov, K., New operations defined over the intuitionistic fuzzy Sets, Fuzzy Sets and Systems, 61(2)(1994), 137-142.

[3] Atanassov, K., Intuitionistic fuzzy sets, Theory and applications, Studies in Fuzziness and Soft computing, 35. Heidelberg, Physica- Verlag1990.

[4] Atanassov. K., Intuitionistic fuzzy systems, Springer Physica-Verlag, Berlin, 1999.

[5] Hung, Y., and Chen, Z., on ideals in BCK-algebras, Math. Japan. 50(1999), no.2,211-226.CMP 1 718 851. Zb1 938.06018.

[6] Jun, Y.B., On n-fold fuzzy positive implicative ideals of BCK-algebras, Int.J.Math.Math.Sci.26 (2001), no.9, 525-537.

[7] Satyanarayana, B., Kondala Rao, E.V., and Krishna, L., Intuitionistic fuzzy BCK-algebra, ANU Journal of Physical Sciences, 1(2)(2009), 21-32.

[8] Satyanarayana, B. and Durga Prasad, R., Product of intuitionistic Fuzzy BCK- algebras, Advances in Fuzzy Mathematics, 4(1)(2009), 1-8, Research India Publication.

[9] Satyanarayana, B., and Durga Prasad, R., Direct product of finite Intuitionist fuzzy BCK-algebras, Global Journal of Pure and Applied Mathematics, Research India Publication, 5(2)(2009), 125-138.

[10] Satyanarayana, B., Bindu Madhavi.U and Durga Prasad, R., On Intuitionistic fuzzy H-ideals in BCK-algebras, International Journal of Algebra, 4(15)(2010), 743-749.

[11] Satyanarayana, B., Bindu Madhavi.U., and Durga Prasad, R., on foldness of Intuitionistic fuzzy H-ideals in BCK-algebras, International Mathematical forum, 5(45) (2010), 2205-2211.

[12] Satyanarayana, B., and Durga Prasad, R., On Intuitionistic fuzzy ideals in BCK- algebras, International J. of Math. Sci and Appls, 5(1) (2011), 283-294.

[13] Satyanarayana, B., and Durga Prasad, R., on foldness of Intuitionistic fuzzy implicative / Commutative ideals of BCK-algebras (Communicated).

[14] Zadeh, L.A., Fuzzy sets, Information Control, 8(1965), 338-353.

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