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PROPERTIES OF LATTICE ORDERED GROUPS AND ORDERED Γ- SEMIGROUPS

VB Subrahmanyeswara Rao Seetamraju*
Assistant Professor, VKR, VNB & AGK College of Engineering, Gudivada, Andhra Pradesh, India

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ABSTRACT

In this paper, we present the properties of lattice ordered groups derived from the properties of partially ordered groups. The notion of Γ-semi groups was introduced by Sen in 1981. The concept of Γ- semigroups is a generalization of the concept of semigroups Many classical notions of semigroups have been extended to Γ-semigroups, (S, Γ , ≤) is called an ordered Γ-semigroup if S is a Γ-semigroup and (S, ≤) is a partially ordered set such that $a \leq b \Rightarrow a\,\gamma\,c \leq b\,\gamma\,c$ *and* $c\,\gamma\,a \leq c\,\gamma\,b$ *for all* $a, b, c\, \epsilon\, S$ *and* $\gamma\, \epsilon\, \Gamma$

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1. INTRODUCTION

A semi group S is said to be partially ordered or a partially ordered semi group if it is associated with a partial ordering ≤ which is defined by $\boldsymbol{a \leq b}$ implies that $\boldsymbol{xay \leq xby}$ for all $\boldsymbol{x, y}$ in S. The natural partial order which is an obvious partial ordering defined by $\boldsymbol{a \leq b}$ if and only if $\boldsymbol{a = cb}$ for some $\boldsymbol{c = c^2 \epsilon S}$. This natural partial ordering is compatible with multiplication. Some of the basic properties and results were given by Donald B. McAlister and some of the foundational results are due to A.H. Clifford. Suppose G is a partially ordered group i.e., G is a group partially ordered by ≤. Now $\boldsymbol{a \leq b}$ if and only if $\boldsymbol{1 \leq a^{-1}b}$ or equivalently $\boldsymbol{a \leq b}$ if and only if $\boldsymbol{1 \leq ba^{-1}}$

2. PROPERTIES OF PARTIALLY ORDERED, LATTICE ORDERED GROUPS

Here we consider the set G^+ consisting of elements exceeding the identity 1 and has the following properties:
(i) G^+ is a submonoid of G
(ii) a $G^+ = G^+a$ for each a *in* G
(iii) 1 is the only invertible element of G^+

Notation 2.1: *Let G be a partially ordered group and suppose that* $a, b \in G$. *Then the least upper bound of a and b is denoted by* $a \vee b$, *read as a join b. The greatest lower bound of a and b is denoted by* $a \wedge b$, *read as a meet b.*

The following Proposition is due to Donald B.McAlister [2]

Proposition 2.2: *Let G be a partially ordered group and suppose that* $a, b \in G$. *Then a and b have a least upper bound* $a \vee b$ *in G if and only if they have a greatest lower bound* $a \wedge b$. *This is only possible only when* a^{-1} *and* b^{-1} *have a least upper bound. In particular,*

$$a \wedge b = a(a \vee b)^{-1}b \qquad a \wedge b = (a^{-1} \vee b^{-1})^{-1}$$
$$a \wedge b = a(a \wedge b)^{-1}b \qquad a \vee b = (a^{-1} \wedge b^{-1})^{-1}$$

Also for any g *in* G,

$$g(a \vee b) = ga \vee gb \qquad g(a \wedge b) = ga \wedge gb$$
$$(a \vee b)g = ag \vee bg \qquad (a \wedge b)g = ag \wedge b$$

Corollary 2.3: *The following are equivalent for a partially ordered group G.*
(i) G is a ∨*-semilattice under* ≤
(ii) G is a ∧ *-semilattice under* ≤
(iii) $a \vee 1$ *exists for each* $a \in G$
(iv) $a \wedge 1$ *exists for each* $a \in G$
(v) $a \vee b$ *exists for each* $a, b \in G^+$
(vi) $a \wedge b$ *exists for each* $a, b \in G^+$
(vii) for each $a, b \in G^+$ *there exists* $c \in G^+$ *such that* $G^+a \wedge G^+b = G^+c$

***Corresponding author:** *VB Subrahmanyeswara Rao Seetamraju**
Assistant Professor, VKR, VNB & AGK College of Engineering, Gudivada, Andhra Pradesh, India

If G satisfies one of the conditions in the above corollary, we say that G is a lattice ordered group or simply latticed group.

Definition 2.4: A lattice G is called a distributive lattice, if for any $a, b, c \in G$
(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and (ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
Clearly conditions (i) and (ii) are equivalent.

Definition 2.5: A lattice G is called a modular lattice, if for any $a, b, c \in G$ such that $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Theorem 2.6: *Let G be a lattice ordered group under a partial order $\leq$, then G is a modular lattice under $\leq$.*

Proof: The proof is obvious, because if G is a lattice ordered group with respect to a partial order $\leq$, then G is a distributive lattice under $\leq$ and every distributive lattice is a modular lattice.

Definition 2.7: Two elements a, b of a lattice ordered group G are said to be orthogonal if $\mathbf{a} \wedge \mathbf{b} = \mathbf{1}$

Proposition 2.8: *Let G be a lattice ordered group and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{G}$. If $\boldsymbol{a} \wedge \boldsymbol{b} = \mathbf{1}$, then $\boldsymbol{ac} \wedge \boldsymbol{bc} = \boldsymbol{c}$.*

Proof: Since $a \wedge b = 1$,

$$\begin{aligned} ac \wedge bc &= 1\,(ac \wedge bc) \\ &= (a \wedge b)(ac \wedge bc) \\ &= a\,(ac \wedge bc) \wedge b\,(ac \wedge bc) \\ &= a^2c \wedge abc \wedge bac \wedge b^2 c \\ &= a^2c \wedge abc \wedge b^2 c, \text{since } a \wedge a = a \\ &= c\,(a(a \wedge b) \wedge b^2) \\ &= c\,(a \wedge b^2) \\ &= c, \text{since } a^m \wedge b^n = 1\ \forall\, m, n \geq 0 \end{aligned}$$

Definition 2.9: A homomorphism θ of a lattice ordered group G into a lattice ordered group H is an l- homomorphism if θ respects $\vee$ and i.e., $(\mathbf{a} \vee \mathbf{b})\,\boldsymbol{\theta} = \mathbf{a}\boldsymbol{\theta} \vee \mathbf{b}\boldsymbol{\theta}$ and $(\mathbf{a} \wedge \mathbf{b})\,\boldsymbol{\theta} = \mathbf{a}\boldsymbol{\theta} \wedge \mathbf{b}\boldsymbol{\theta}$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{G}$.

Definition 2.10: Let M and **Γ** be any two non-empty sets. If there exists a mapping $\mathbf{M} \times \boldsymbol{\Gamma} \times \mathbf{M} \to \mathbf{M}$, written $(\mathbf{a}, \boldsymbol{\gamma}, \mathbf{b})$ by $\mathbf{a}\boldsymbol{\gamma}\mathbf{b}$, M is called a **Γ**-Semigroup if M satisfies the identity $(\mathbf{a}\boldsymbol{\gamma}\mathbf{b})\boldsymbol{\mu}\mathbf{c} = \mathbf{a}\boldsymbol{\gamma}(\mathbf{b}\boldsymbol{\mu}\mathbf{c})$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{M}$ **and** $\boldsymbol{\gamma}, \boldsymbol{\mu} \in \boldsymbol{\Gamma}$

Notation 2.11: *Green's relations are defined on a **Γ**-Semigroup S as follows:*
$\boldsymbol{a\mathcal{L}b}$ ***if and only if*** $\boldsymbol{S'\Gamma a = S'\Gamma b}$
$\boldsymbol{a\mathcal{R}b}$ ***if and only if*** $\boldsymbol{a\Gamma S' = b\Gamma S'}$
$\boldsymbol{a\mathcal{D}b}$ ***if and only if*** $\boldsymbol{a\Gamma\mathcal{L}\ o\ \mathcal{R}\Gamma b}$
Where $\boldsymbol{S'}$ *denotes S if S has an identity and* $\boldsymbol{S' = S \cup \{1\}}$ *where 1 acts like the identity and* $\boldsymbol{\mathcal{D} = \mathcal{R}\ o\ \mathcal{L}}$

Definition 2.12: A semi group S is regular if and only if, for each $\mathbf{a} \in \mathbf{S}$, there exists $\mathbf{x} \in \mathbf{S}$ such that $\mathbf{a} = \mathbf{a}\boldsymbol{\alpha}\mathbf{x}\boldsymbol{\alpha}\mathbf{a}$ for $\boldsymbol{\alpha} \in \boldsymbol{\Gamma}$. In this case $\mathbf{a}' = \mathbf{xax}$ is a solution to the pair of equations $\mathbf{a} = \mathbf{aya}$, $\mathbf{y} = \mathbf{yay}$. The element a is called an inverse for $\mathbf{a}$ in S and the semi group is said to be an Inverse **Γ**- semi group if each element of S has a unique inverse in S.

3. GENERAL PROPERTIES OF Γ-SEMIGROUPS

Lemma 3.1: *For every* $\boldsymbol{\Gamma} -$ *semigroup S, the following hold*
i) $\boldsymbol{a \leq e, a \in S, e \in E_s}$ ***imply*** $\boldsymbol{a \in E_s}$
ii) $\boldsymbol{a \leq b, b \in S, b}$ ***regular imply a regular***
iii) $\boldsymbol{a \leq b, a, b \in S, a}$ ***regular imply*** $\boldsymbol{a = eb = bf, e, f \in E_s, e\,\mathcal{D}\,f}$

Proof: i) For $\mathbf{a} \in \mathbf{S}$, **we have** $\mathbf{a} = \mathbf{x}\boldsymbol{\alpha}\mathbf{e} = \mathbf{e}\boldsymbol{\alpha}\mathbf{y}$ and $\mathbf{x}\boldsymbol{\alpha}\mathbf{a} = \mathbf{a} = \mathbf{a}\boldsymbol{\alpha}\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbf{S}', \boldsymbol{\alpha} \in \boldsymbol{\Gamma}$
Now, $\mathbf{a^2} = \mathbf{a}\boldsymbol{\alpha}\mathbf{a}, \boldsymbol{\alpha} \in \boldsymbol{\Gamma}$
$\qquad = \mathbf{x}\boldsymbol{\alpha}\mathbf{e}\ \boldsymbol{\alpha}\ \mathbf{e}\boldsymbol{\alpha}\mathbf{y}$

$= \mathbf{x\alpha e\ \alpha\ y}$
$= \mathbf{a\alpha y}$
$= \mathbf{a}$
$\therefore \mathbf{a} \in \mathbf{E_s}$

ii) Suppose $\mathbf{a \le b}$, $\mathbf{b}$ **regular then** $\mathbf{a = x\alpha b = b\alpha y;\ \alpha = a\alpha y}$

But for every inverse element $\mathbf{b'}$ **of** $\mathbf{b, a = a\alpha y = x\alpha b\alpha y}$

$$= \mathbf{x\alpha(b\alpha b'\alpha b)\alpha y}$$
$$= \mathbf{(x\alpha b)\alpha b'\alpha b\alpha y}$$
$$= \mathbf{a\alpha b'\alpha a}$$

Hence **a is regular**.

iii) Suppose $a \le b$, a is regular, which implies $a = x\alpha b = b\alpha y$ and
$x\alpha a = a = a\alpha y (x, y \in S')$
But For every inverse element a' of a, $a = a\alpha a'\alpha a = a\alpha a'\alpha\, x\alpha b = e\alpha b$ for $e = a\alpha a'\alpha x$

Similarly, $a = b\alpha f$ for $f = y\alpha a'\beta a \in E_s$

Finally, since $e = a\alpha a'\beta x, a = e\alpha b$

$\therefore e\, \mathcal{R}\, a$

Since $a = b\alpha f, f = y\alpha a'\beta a$, we have $a\, \mathcal{L}\, f$

Hence $e\, \mathcal{D}\, f\ (\because \mathcal{D} = \mathcal{R} \circ \mathcal{L})$

Definition 3.2: $(S, \Gamma, \le)$ is called an ordered $\Gamma -$ semigroup if (S, Γ) is a $\Gamma -$semigroup and $(S, \le)$ is a partially ordered set such that
$a \le b \implies a\,\gamma\, c \le b\,\gamma\, c$ and $c\,\gamma a \le c\,\gamma b$, for all $a, b, c \in S$ and $\gamma \in \Gamma$

The above property is similar to natural partial order compatibility with multiplication $a \le b \implies ac \le bc$ and $ca \le cb$ for all $c \in S$

Definition 3.3: The idempotents E_s of a Γ-semigroup can be given a partial order, ($E_s \ne \emptyset$) as follows:
For $e, f \in E_s$ define $e \le f \Leftrightarrow e\alpha f = e = f\alpha e$

Lemma 3.4: *The relation ≤ is a partial order on E_s such that for $e, f \in E_s$ $e \le f \Leftrightarrow e\alpha f = e = f\alpha e$*

Proof: Firstly, for all $e \in E_s$, $e\alpha e = e$
Therefore $e \le e$
$\therefore \le$ is Reflexive

If $e \le f$ and $f \le e$, then $e = e\alpha f = f$

Therefore $\le$ is Anti-Symmetric

If $e \le f$ and $f \le h$, then
$e = e\alpha f = e\alpha f\beta h = e\alpha h$ and $e = f\alpha e = h\alpha f\alpha e = h\alpha e$

Therefore $e \le h$

Thus $\le$ is Transitive and hence the relation ≤ is a partial order on E_s

Definition 3.5: If S is a commutative $\Gamma -$ semigroup and all its elements are idempotents ($S = E_s$) then S is called a $\Gamma -$semilattice.

Hence for all $x, y \in S$: $x\alpha x = x$ and $x\alpha y = y\alpha x$

The relation ≤ on idempotents is defined on the whole of a $\Gamma -$ semilattice 'S'.

An element g is a lower bound of elements e and f if $g \leq e$ and $g \leq f$ i.e.,
$g\alpha e = g = e\alpha g$ and $g\alpha f = g = f\alpha g$

Lemma 3.6: *Let S be a* Γ *−semilattice. Then* $e\alpha f \epsilon S$ *is the greatest lower bound of the elements e and f of S*

Proof: Let $e\alpha f \epsilon S$, $\alpha \in \Gamma$. Then $e\alpha f = f\alpha e = f\alpha f\alpha e = e\alpha f\alpha f$

$\therefore (e\alpha f)\alpha f = e\alpha f = f\alpha(e\alpha f)$

This means $e\alpha f \leq f$

So, $(e\alpha f)\alpha e = f\alpha e\alpha e = f\alpha e = e\alpha f$

Therefore, $e\alpha(e\alpha f) = e\alpha(f\alpha e)$
$= e\alpha f\alpha e$
$= f\alpha e\alpha e$
$= f\alpha e$
$= e\alpha f$
$\therefore e\alpha f \leq e$

Thus $e\alpha f$ is a lower bound of e and f

If g is a lower bound of e and f, then

$g = g\alpha f = g\alpha e$, and therefore $g\alpha(e\alpha f) = (g\alpha e)\alpha f = g\alpha f = g$ And hence $g \leq e\alpha f$

So, $e\alpha f \epsilon S$ is the greatest lower bound of the elements e and f

4. PARTIAL ORDERING IN INVERSE Γ −SEMI GROUPS

In any Γ −semigroup S, the idempotent can be partially ordered by the relation
$e \leq f \iff e\alpha f = e = f\alpha e \; \forall \; e, f \epsilon S, \alpha \epsilon \Gamma$

Analogous to this partial ordering, we can define a partial ordering to all the elements of an inverse Γ −semigroup by
$x \leq y \iff \exists \; e \epsilon E_s : x = e\alpha y$

Since $= (x \, \alpha \, x^{-1})\alpha x$, $x\alpha x^{-1} \epsilon E_s$, we have $\leq$ is reflexive

If $x = e\alpha y$ and $y = f\alpha x$, then

$x = e\alpha y = e\alpha(e\alpha y) = e\alpha x$

Hence $x = e\alpha y = e\alpha f\alpha x = f\alpha x = y$, so $\leq$ is antisymmetric

If $x = e\alpha y$ and $y = f\alpha z$, then

$x = e\alpha y = e\alpha f\alpha z, e\alpha f \epsilon E_s$, thus $\leq$ is transitive

Lemma 4.1: *In an inverse* $\Gamma -$ *semigroup S,* $x \leq y \iff x = x\alpha x^{-1}\alpha y$

Proof: $x \leq y \iff \exists e \epsilon E_s : x = y\alpha e$
$\iff x\alpha x^{-1} = y\alpha x^{-1}$
$\iff x = x\alpha y^{-1}\alpha x$
$\iff x\alpha x^{-1} = x\alpha y^{-1}$
$\iff x^{-1}\alpha x = y^{-1}\alpha \, x$
$\iff x = x\alpha x^{-1}\alpha y$

Definition 4.2: Let S be an inverse Γ- semigrou. Then S is E-Unitary if and only if eαe = e = eαa implies aαa = a for all α ∈ Γ

Lemma 4.3: *Let S be an E-Unitary inverse* $\Gamma -$ *semigroup. Then* $(x, y) \in \sigma$ *if and only if* $x\alpha x^{-1}\alpha y = y\alpha y^{-1}\alpha x$, *σ is an equivalence on S*

Proof: Suppose $x\alpha x^{-1}\alpha y = y\alpha y^{-1}\alpha x$. Then $e\alpha y = e\alpha x$ for $x, y \epsilon S$ and $\alpha \epsilon \Gamma$

$\Rightarrow (x, y) \in \sigma$

Conversely Suppose that $(x, y) \in \sigma$. Then $(x^{-1}, y^{-1}) \in \sigma$

$\Rightarrow e\alpha x^{-1} = e\alpha y^{-1}$ for some $e \in E_s$

$\Rightarrow e\alpha x^{-1}\alpha y = e\alpha y^{-1}\alpha y$ is idempotent

Since S is E-Unitary, $x^{-1}\alpha y$ is idempotent.

$\therefore\ x^{-1}\alpha y = (x^{-1}\alpha y)\alpha(x^{-1}\alpha y)^{-1}$

$= x^{-1}\alpha y\ \alpha\ y^{-1}\alpha x$

$\Rightarrow x\alpha x^{-1}\alpha y = x\alpha x^{-1}\alpha y\alpha y^{-1}\alpha x = (y\alpha y^{-1})\alpha(x\alpha x^{-1}\alpha x)$

$= y\alpha y^{-1}\alpha x$, Since idempotents commute.

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