

$(\tau_1, \tau_2)^*$ - Q^* HOMEOMORPHISM IN BITOPOLOGICAL SPACES

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ABSTRACT

Aim of this paper is to introduce and study the new type of homeomorphism, namely $(\tau_1, \tau_2)^$ - Q^* homeomorphism and $\tau_1\tau_2$ - Q^* homeomorphism in bitopological spaces. Also we define $(\tau_1, \tau_2)^*$ -irreducible spaces and $\tau_1\tau_2$ - irreducible spaces. Here researchers proved that the set of all $(\tau_1, \tau_2)^*$ - Q^* homeomorphism forms a group.*

Keywords: $(\tau_1, \tau_2)^*$ - Q^* homeomorphism, $\tau_1\tau_2$ - Q^* homeomorphism, $(\tau_1, \tau_2)^*$ - irreducible spaces and $\tau_1\tau_2$ -irreducible spaces.

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1. INTRODUCTION

The notion of homeomorphism plays a dominant role in topology and so many authors introduced various types of homeomorphisms in topological spaces. In 1995, Maki, Devi and Balachandran [3] introduced the concepts of semi-generalized homeomorphisms and generalized semi-homeomorphisms and studied some semi topological properties. Devi and Balachandran introduced a generalization of α -homeomorphism in 2001.

Recently, P. Padma and S. Udaykumar [8] introduced and studied the concept of $(\tau_1, \tau_2)^*$ - Q^* continuous maps in bitopological spaces.

The purpose of this paper is to introduce the concepts of homeomorphisms by using $(\tau_1, \tau_2)^*$ - Q^* open sets. In this paper, we introduce the concepts $\tau_1\tau_2$ - Q^* homeomorphism, and $(\tau_1, \tau_2)^*$ - Q^* homeomorphism and investigate their basic properties. Also we define and studied the properties of $(\tau_1, \tau_2)^*$ -irreducible spaces and $\tau_1\tau_2$ - irreducible spaces.

The most important property is that the set of all $(\tau_1, \tau_2)^*$ - Q^* homeomorphisms is a group under composition of functions.

2. PRELIMINARIES

Throughout this paper X and Y always represent nonempty bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) . Now we shall require the following known definitions are prerequisites.

Definition 2.1: A subset S of X is called $(\tau_1, \tau_2)^*$ -open if $S \in \tau_1 \cup \tau_2$ and the complement of $(\tau_1, \tau_2)^*$ - open set is $(\tau_1, \tau_2)^*$ - closed.

Definition 2.2: A map $f: X \rightarrow Y$ is called $(\tau_1, \tau_2)^*$ - Q^* continuous if the inverse image of each $(\sigma_1, \sigma_2)^*$ - Q^* closed in Y is $\tau_1\tau_2$ - closed in X .

Definition 2.3 [6]: A map $f: X \rightarrow Y$ is called $\tau_1\tau_2$ - Q^* - continuous if the inverse image of each $\sigma_1\sigma_2$ - Q^* closed in Y is τ_2 - closed in X .

Definition 2.4: A subset S of X is said to be $(\tau_1, \tau_2)^*$ - semi open set if $S \subseteq \tau_1\tau_2 \text{ cl}(\tau_1\tau_2 - \text{int}(S))$. The complement of $(\tau_1, \tau_2)^*$ - semi open set is $(\tau_1, \tau_2)^*$ - semi closed.

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3. $(\tau_1, \tau_2)^*$ - Q^* HOMEOMORPHISM

Throughout this paper X and Y always represent nonempty bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) . We denote the family of all $(\tau_1, \tau_2)^*$ - Q^* homeomorphisms from (X, τ_1, τ_2) onto itself by $(\tau_1, \tau_2)^*$ - Q^* $H(X)$ and the family of all $(\tau_1, \tau_2)^*$ - closed set in (X, τ_1, τ_2) is denoted by $(\tau_1, \tau_2)^*$ - $C(X)$.

Definition 3.1[8]: A bijection $f: X \rightarrow Y$ is called $(\tau_1, \tau_2)^*$ - Q^* homeomorphism, if f is $(\tau_1, \tau_2)^*$ - Q^* continuous and its inverse also $(\tau_1, \tau_2)^*$ - Q^* continuous.

Example 3.1: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$, $\tau_2 = \{\phi, X, \{b\}, \{b, c\}\}$ and $\sigma_1 = \{\phi, Y, \{b\}\}$, $\sigma_2 = \{\phi, Y, \{b\}, \{b, c\}\}$. Then $\phi, \{a\}, \{a, c\}$ are $(\sigma_1, \sigma_2)^*$ - Q^* closed in Y . Let $f: X \rightarrow Y$ be the identity map. Then $f(\phi) = \phi$, $f(\{a, c\}) = \{a, c\}$, $f(\{a\}) = \{a\}$. Since $\phi, \{a, c\}, \{a\}$ are $\tau_1\tau_2$ - closed in X . Therefore, f and f^{-1} are $(\tau_1, \tau_2)^*$ - Q^* continuous. Hence f is $(\tau_1, \tau_2)^*$ - Q^* homeomorphism.

Definition 3.2: A subset S of X is called pairwise $(\tau_1, \tau_2)^*$ - Q^* homeomorphism in X if S is both $(\tau_1, \tau_2)^*$ - Q^* homeomorphism and $(\tau_2, \tau_1)^*$ - Q^* homeomorphism.

Definition 3.3: A space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)^*$ - Q^* space if every $(\tau_1, \tau_2)^*$ - Q^* closed is $(\tau_1, \tau_2)^*$ - closed.

Proposition 3.1: If $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ are $(\tau_1, \tau_2)^*$ - Q^* homeomorphisms, then $g \circ f: (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is also a $(\tau_1, \tau_2)^*$ - Q^* homeomorphism.

Proof: Let U be a $(\eta_1, \eta_2)^*$ - Q^* open set in (Z, η_1, η_2) .

Now $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$.

By hypothesis, V is $(\sigma_1, \sigma_2)^*$ - Q^* open in (Y, σ_1, σ_2) and again by hypothesis, $f^{-1}(V)$ is $(\tau_1, \tau_2)^*$ - Q^* open in (X, τ_1, τ_2) .

Therefore, $(g \circ f)$ is $(\tau_1, \tau_2)^*$ - Q^* continuous.

Also for a $(\tau_1, \tau_2)^*$ - Q^* open set G in (X, τ_1, τ_2) ,

We have $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$.

By hypothesis,

$f(G)$ is $(\sigma_1, \sigma_2)^*$ - Q^* open in (Y, σ_1, σ_2) and again by hypothesis, $g(W)$ is $(\eta_1, \eta_2)^*$ - Q^* open in (Z, η_1, η_2) .

Therefore, $(g \circ f)^{-1}$ is $(\tau_1, \tau_2)^*$ - Q^* continuous.

Hence $g \circ f$ is $(\tau_1, \tau_2)^*$ - Q^* homeomorphism.

Example 3.2: Let $X = Y = Z = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$, $\tau_2 = \{\phi, X, \{b\}, \{b, c\}\}$ and $\sigma_1 = \{\phi, Y, \{b\}\}$, $\sigma_2 = \{\phi, Y, \{b\}, \{b, c\}\}$, $\eta_1 = \{\phi, Z, \{b\}, \{b, c\}\}$, $\eta_2 = \{\phi, Z, \{b, c\}\}$. Then $\phi, \{a\}, \{a, c\}$ are $(\sigma_1, \sigma_2)^*$ - Q^* closed in Y . Let $f: X \rightarrow Y$ be the identity map. Then f and g are $(\tau_1, \tau_2)^*$ - Q^* homeomorphism. Here $g \circ f$ is $(\tau_1, \tau_2)^*$ - Q^* continuous, since $\{b, c\}$ is $(\eta_1, \eta_2)^*$ - Q^* open in Z and $(g \circ f)^{-1}(\{b, c\}) = \{b, c\}$ is $(\tau_1, \tau_2)^*$ - Q^* open in (X, τ_1, τ_2) . Hence $g \circ f$ is $(\tau_1, \tau_2)^*$ - Q^* homeomorphism.

Proposition 3.2: The set $(\tau_1, \tau_2)^*$ - Q^* $H(X)$ is a group.

Proof: Define $\Psi: (\tau_1, \tau_2)^*$ - Q^* $H(X) \times (\tau_1, \tau_2)^*$ - Q^* $H(X) \rightarrow (\tau_1, \tau_2)^*$ - Q^* (X) by $\Psi(f, g) = (g \circ f)$ for every $f, g \in (\tau_1, \tau_2)^*$ - Q^* $H(X)$.

Then by proposition 3.1, $(g \circ f) \in (\tau_1, \tau_2)^*$ - Q^* (X) .

Hence $(\tau_1, \tau_2)^*$ - Q^* $H(X)$ is closed.

We know that the composition of maps is associative.

The identity map

$i: (X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2)$ is a $(\tau_1, \tau_2)^*$ - Q^* homeomorphism and $i \in (\tau_1, \tau_2)^*$ - Q^* $H(X)$.

Also $i \circ f = f \circ i = f$ for every $f \in (\tau_1, \tau_2)^* - Q^* H(X)$.

For any $f \in (\tau_1, \tau_2)^* - Q^* H(X)$,

$$f \circ f^{-1} = f^{-1} \circ f = i.$$

Hence inverse exists for each element of $(\tau_1, \tau_2)^* - Q^* H(X)$.

Thus, $(\tau_1, \tau_2)^* - Q^* H(X)$ is a group under composition of maps.

Theorem 3.1: Every $(\tau_1, \tau_2)^* - Q^*$ homeomorphism is a $(\tau_1, \tau_2)^*$ - homeomorphism.

Proof: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Then f is bijective and both f and f^{-1} are $(\tau_1, \tau_2)^* - Q^*$ continuous.

Since every $(\tau_1, \tau_2)^* - Q^*$ continuous function is $(\tau_1, \tau_2)^*$ continuous we have f and f^{-1} are $(\tau_1, \tau_2)^*$ - continuous.

This shows that f is a $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Remark 3.1: The converse of the above theorem need not be true, as shown in the following example.

Example 3.3: In example 3.1, f is $(\tau_1, \tau_2)^* - Q^*$ homeomorphism but not $(\tau_1, \tau_2)^*$ - homeomorphism.

Proposition 3.3: If $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ are $(\tau_1, \tau_2)^* - Q^*$ homeomorphisms, then $g \circ f: (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is also a $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Proof: Let U be a $\eta_1 \eta_2 - Q^*$ open set in (Z, η_1, η_2) .

$$\text{Now } (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V), \text{ where } V = g^{-1}(U).$$

By hypothesis, V is $\sigma_1 \sigma_2 - Q^*$ open in (Y, σ_1, σ_2) and again by hypothesis, $f^{-1}(V)$ is $\tau_1 \tau_2 - Q^*$ open in (X, τ_1, τ_2) .

Therefore, $(g \circ f)$ is $\tau_1 \tau_2 - Q^*$ continuous.

Also for a $\tau_1 \tau_2 - Q^*$ open set G in (X, τ_1, τ_2) ,

$$\text{We have } (g \circ f)(G) = g(f(G)) = g(W), \text{ where } W = f(G).$$

By hypothesis,

$f(G)$ is $\sigma_1 \sigma_2 - Q^*$ open in (Y, σ_1, σ_2) and again by hypothesis, $g(W)$ is $\eta_1 \eta_2 - Q^*$ open in (Z, η_1, η_2) .

Therefore, $(g \circ f)^{-1}$ is $\tau_1 \tau_2 - Q^*$ continuous.

Hence $g \circ f$ is $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Example 3.4: Let $X = Y = Z = \{a, b, c\}$, $\tau_1 = \{\phi, X\}$, $\tau_2 = \{\phi, X, \{a\}, \{a, c\}\}$ and $\sigma_1 = \{\phi, Y, \{a\}\}$, $\sigma_2 = \{\phi, Y, \{a\}, \{a, c\}\}$, $\eta_1 = \{\phi, Z, \{a\}, \{a, c\}\}$, $\eta_2 = \{\phi, Z, \{a, c\}\}$. Then $\phi, \{b, c\}, \{b\}$ are $\sigma_1 \sigma_2 - Q^*$ closed in Y . Let $f: X \rightarrow Y$ be the identity map. Then f and g are $(\tau_1, \tau_2)^* - Q^*$ homeomorphism. Here $g \circ f$ is $\tau_1 \tau_2 - Q^*$ continuous, since $\{a, c\}$ is $\eta_1 \eta_2 - Q^*$ open in Z and $(g \circ f)^{-1}(\{a, c\}) = \{a, c\}$ is $\tau_1 \tau_2 - Q^*$ open in (X, τ_1, τ_2) . Hence $g \circ f$ is $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Proposition 3.4: The set $\tau_1 \tau_2 - Q^* H(X, \tau_1, \tau_2)$ is a group.

Proof: Define $\Psi: \tau_1 \tau_2 - Q^* H(X, \tau_1, \tau_2) \times \tau_1 \tau_2 - Q^* H(X, \tau_1, \tau_2) \rightarrow \tau_1 \tau_2 - Q^* H(X, \tau_1, \tau_2)$ by $\Psi(f, g) = (g \circ f)$ for every $f, g \in \tau_1 \tau_2 - Q^* H(X, \tau_1, \tau_2)$.

Then by proposition 3.3, $(g \circ f) \in \tau_1 \tau_2 - Q^* H(X, \tau_1, \tau_2)$.

Hence $\tau_1 \tau_2 - Q^* H(X, \tau_1, \tau_2)$ is closed.

We know that the composition of maps is associative.

The identity map

$i: (X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2)$ is a $\tau_1 \tau_2 - Q^*$ homeomorphism and $i \in \tau_1 \tau_2 - Q^*H(X, \tau_1, \tau_2)$.

Also $i \circ f = f \circ i = f$ for every $f \in \tau_1 \tau_2 - Q^*H(X, \tau_1, \tau_2)$.

For any $f \in \tau_1 \tau_2 - Q^*H(X, \tau_1, \tau_2)$,

$$f \circ f^{-1} = f^{-1} \circ f = i.$$

Hence inverse exists for each element of $\tau_1 \tau_2 - Q^*H(X, \tau_1, \tau_2)$.

Thus, $\tau_1 \tau_2 - Q^*H(X, \tau_1, \tau_2)$ is a group under composition of maps.

Theorem 3.2 -Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map. Then the following are true.

- i) Every $\tau_1 \tau_2 - Q^*$ homeomorphism is $\tau_1 \tau_2$ - homeomorphism.
- ii) Every $(\tau_1, \tau_2)^* - Q^*$ homeomorphism is $\tau_1 \tau_2 - Q^*$ homeomorphism.

Proof: The proof is obvious.

Definition 3.4: For a subset A of a space (X, τ_1, τ_2) we define the **$(\tau_1, \tau_2)^* - Q^*$ kernel** of A (briefly, $(\tau_1, \tau_2)^* - Q^*$ $\ker(A)$) as follows: $(\tau_1, \tau_2)^* - Q^* \ker(A) = \bigcap \{F: F \in (\tau_1, \tau_2)^* - Q^* O(X, \tau_1, \tau_2); A \subset F\}$. A is said to be a **$(\tau_1, \tau_2)^* - Q^*$ - Λ set** in (X, τ_1, τ_2) if $A = (\tau_1, \tau_2)^* - Q^* \ker(A)$, or equivalently, if A is the intersection of $(\tau_1, \tau_2)^* - Q^*$ open sets. A is said to be **$(\tau_1, \tau_2)^* - Q^*$ λ - closed** in (X, τ_1, τ_2) if it is the intersection of a $(\tau_1, \tau_2)^* - Q^* - \Lambda$ set in (X, τ_1, τ_2) and a quasi closed set in (X, τ_1, τ_2) . Clearly, $(\tau_1, \tau_2)^* - Q^* - \Lambda$ set and $(\tau_1, \tau_2)^* - Q^*$ closed sets are $(\tau_1, \tau_2)^* - Q^* - \lambda$ closed; complements of $(\tau_1, \tau_2)^* - Q^* - \lambda$ closed sets in (X, τ_1, τ_2) are said to be $(\tau_1, \tau_2)^* - Q^* - \lambda$ open in (X, τ_1, τ_2) .

Proposition 3.5: For a subset A of a space (X, τ_1, τ_2) , the following are equivalent:

- (i) A is $(\tau_1, \tau_2)^* - Q^* - \lambda$ closed in (X, τ_1, τ_2) .
- (ii) $A = L \cap (\tau_1, \tau_2)^* - Q^* \text{cl}(A)$, where L is a $(\tau_1, \tau_2)^* - Q^* - \Lambda$ set in (X, τ_1, τ_2) .
- (iii) $A = (\tau_1, \tau_2)^* - Q^* \ker(A) \cap (\tau_1, \tau_2)^* - Q^* \text{cl}(A)$.

Definition 3.5: A bijection $f: X \rightarrow Y$ is called **$(\tau_1, \tau_2)^* - Q^*$ homeomorphism**, if f is $(\tau_1, \tau_2)^* - Q^*$ irresolute and its inverse also $(\tau_1, \tau_2)^* - Q^*$ irresolute.

Remark 3.2: We say that spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are $(\tau_1, \tau_2)^* - Q^*$ homeomorphic if there exists a $(\tau_1, \tau_2)^* - Q^*$ homeomorphism from (X, τ_1, τ_2) onto (Y, σ_1, σ_2) .

Theorem 3.3: Every $(\tau_1, \tau_2)^* - Q^*$ homeomorphism is a $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Proof: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Then f is bijective, $(\tau_1, \tau_2)^* - Q^*$ irresolute and f^{-1} is $(\tau_1, \tau_2)^* - Q^*$ irresolute.

Since every $(\tau_1, \tau_2)^* - Q^*$ irresolute is $(\tau_1, \tau_2)^* - Q^*$ continuous, f and f^{-1} are $(\tau_1, \tau_2)^* - Q^*$ continuous and so f is a $(\tau_1, \tau_2)^* - Q^*$ homeomorphism.

Remark 3.3: The following example shows that the converse of the above theorem need not be true.

Example 3.5: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}\}$. Clearly $\{b\}$ is $(\tau_1, \tau_2)^* - Q^*$ closed in X . Let $\sigma_1 = \{\phi, Y, \{a\}\}$ and $\sigma_2 = \{\phi, Y\}$.

Then $\sigma_1 \sigma_2$ - open sets on Y are $\phi, Y, \{a\}$ and $\sigma_1 \sigma_2$ - closed sets on X are $\phi, Y, \{b, c\}$. Since $\{b, c\}$ is $(\sigma_1, \sigma_2)^* - Q^*$ closed in Y but $f^{-1}(\{b, c\}) = \{b, c\}$ is not $(\tau_1, \tau_2)^* - Q^*$ open in X and so f is not $(\tau_1, \tau_2)^* - Q^*$ irresolute.

Remark 3.4: The above example shows that the concepts of $(\tau_1, \tau_2)^*$ -homeomorphisms and $(\tau_1, \tau_2)^* - Q^*$ homeomorphism are independent.

Definition 3.6 -A map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(\tau_1, \tau_2)^*$ - Q^* closed if for every $(\tau_1, \tau_2)^*$ - Q^* closed F of X , $f(F)$ is $(\tau_1, \tau_2)^*$ - closed in Y .

Proposition 3.6 - For any bijection $f: X \rightarrow Y$, the following statements are equivalent.

- (a) $f^{-1}: Y \rightarrow X$ is $(\tau_1, \tau_2)^*$ - Q^* continuous.
- (b) f is a $(\tau_1, \tau_2)^*$ - Q^* open map.
- (c) f is a $(\tau_1, \tau_2)^*$ - Q^* closed map.

Proof:

Step 1: (a) \Rightarrow (b)

Let V be a $(\tau_1, \tau_2)^*$ - open set in X .

Then $X - V$ is $(\tau_1, \tau_2)^*$ - closed in X .

Since f^{-1} is $(\tau_1, \tau_2)^*$ - Q^* continuous,

$(f^{-1})^{-1}(X - V) = f(X - V) = Y - f(V)$ is $(\tau_1, \tau_2)^*$ - Q^* closed in Y .

Then $f(V)$ is $(\tau_1, \tau_2)^*$ - Q^* open in Y .

Hence f is a $(\tau_1, \tau_2)^*$ - Q^* open map.

Step 2: (b) \Rightarrow (c).

Let f be a $(\tau_1, \tau_2)^*$ - Q^* open map.

Let U be a $(\tau_1, \tau_2)^*$ - closed set in X .

Then $X - U$ is $(\tau_1, \tau_2)^*$ - open in X .

Since f is $(\tau_1, \tau_2)^*$ - Q^* open,

$f(X - U) = Y - f(U)$ is $(\tau_1, \tau_2)^*$ - Q^* open in Y .

Then $f(U)$ is $(\tau_1, \tau_2)^*$ - Q^* closed in Y .

Hence f is a $(\tau_1, \tau_2)^*$ - Q^* closed.

Step 3: (c) \Rightarrow (a).

Let V be $(\tau_1, \tau_2)^*$ - Q^* closed set in X .

Since $f: X \rightarrow Y$ is $(\tau_1, \tau_2)^*$ - Q^* closed,

$f(V)$ is $(\tau_1, \tau_2)^*$ - Q^* closed in Y .

That is $(f^{-1})^{-1}(f(V))$ is $(\tau_1, \tau_2)^*$ - Q^* closed in X .

Hence f^{-1} is $(\tau_1, \tau_2)^*$ - Q^* continuous.

Proposition 3.7: Let $f: X \rightarrow Y$ be a bijective and $(\tau_1, \tau_2)^*$ - Q^* continuous map. Then the following statements are equivalent.

- (a) f is a $(\tau_1, \tau_2)^*$ - Q^* open map.
- (b) f is a $(\tau_1, \tau_2)^*$ - Q^* homeomorphism.
- (c) f is a $(\tau_1, \tau_2)^*$ - Q^* closed map.

Proof:

Step 1: (a) \Rightarrow (b).

Given f is bijective, $(\tau_1, \tau_2)^*$ - Q^* continuous map and $(\tau_1, \tau_2)^*$ - Q^* open map. Hence f is $(\tau_1, \tau_2)^*$ - Q^* homeomorphism.

Step 2: (b) \Rightarrow (c).

Let f be a $(\tau_1, \tau_2)^*$ - Q^* homeomorphism.

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Hence f is $(\tau_1, \tau_2)^*$ - Q^* open.

By Proposition 3.6, f is $(\tau_1, \tau_2)^*$ - Q^* closed.

Step 3: (c) \Rightarrow (a)

Follows from Proposition 3.6.

Definition 3.7: Let S be a subset of X . Let $x \in X$. Then x is said to be a $(\tau_1, \tau_2)^*$ - Q^* limit point of S if and only if every $(\tau_1, \tau_2)^*$ - Q^* open set containing x contains at least one point other than x .

Definition 3.8: Let S be a subset of X . Then the set of all $(\tau_1, \tau_2)^*$ - Q^* limit points of S is said to be $(\tau_1, \tau_2)^*$ - Q^* derived set of S and it is denoted by $(\tau_1, \tau_2)^*$ - $D Q^*(S)$.

Theorem 3.4: Let A be a subset of X . Let $(\tau_1, \tau_2)^*$ - $D Q^*(A)$ be the set of all $(\tau_1, \tau_2)^*$ - Q^* limit points of A . Then $(\tau_1, \tau_2)^*$ - Q^* $cl(A) = A \cup (\tau_1, \tau_2)^*$ - $D Q^*(A)$.

Proof: Let $x \in A \cup (\tau_1, \tau_2)^*$ - $D Q^*(A)$.

This implies either $x \in A$ or $x \in (\tau_1, \tau_2)^*$ - $D Q^*(A)$.

If $x \in A$, then $x \in (\tau_1, \tau_2)^*$ - Q^* $cl(A)$.

If $x \in (\tau_1, \tau_2)^*$ - $D Q^*(A)$, then every $(\tau_1, \tau_2)^*$ - Q^* open set contains x will intersect with A .

Therefore, $x \in (\tau_1, \tau_2)^*$ - Q^* $cl(A)$.

This implies $A \cup (\tau_1, \tau_2)^*$ - $D Q^*(A) \subseteq (\tau_1, \tau_2)^*$ - Q^* $cl(A)$.

If $x \in (\tau_1, \tau_2)^*$ - Q^* $cl(A)$, then to prove $x \in A \cup (\tau_1, \tau_2)^*$ - $D Q^*(A)$.

If $x \in A$, then $x \in A \cup (\tau_1, \tau_2)^*$ - $D Q^*(A)$.

If $x \notin A$, since $x \in (\tau_1, \tau_2)^*$ - Q^* $cl(A)$ implies every $(\tau_1, \tau_2)^*$ - Q^* open set of x intersects with A .

Hence $x \in (\tau_1, \tau_2)^*$ - $D Q^*(A)$.

Therefore, $(\tau_1, \tau_2)^*$ - Q^* $cl(A) = A \cup (\tau_1, \tau_2)^*$ - $D Q^*(A)$.

Definition 3.9: Let S be a subset of X . Any point of $(\tau_1, \tau_2)^*$ - Q^* $cl(S)$ is referred to as a $(\tau_1, \tau_2)^*$ - Q^* contact (or adherent) point of S .

Definition 3.10 [6]: A bijection $f: X \rightarrow Y$ is called $\tau_1 \tau_2$ - Q^* **homeomorphism**, if f is $\tau_1 \tau_2$ - Q^* continuous and its inverse also $\tau_1 \tau_2$ - Q^* - continuous.

Example 3.6: In example 3.1, $\phi, \{a\}, \{a, c\}$ are $\sigma_1 \sigma_2$ - Q^* closed in Y . Let $f: X \rightarrow Y$ be the identity map. Then $f(\phi) = \phi, f(\{a, c\}) = \{a, c\}, f(\{a\}) = \{a\}$. Since $\phi, \{a, c\}, \{a\}$ are τ_2 - closed in X . Therefore, f and f^{-1} are $\tau_1 \tau_2$ - Q^* continuous. Hence f is $\tau_1 \tau_2$ - Q^* homeomorphism.

Definition 3.11: A bijection $f: X \rightarrow Y$ is called $\tau_1 \tau_2$ - Q^* **homeomorphism**, if f is $\tau_1 \tau_2$ - Q^* irresolute and its inverse also $\tau_1 \tau_2$ - Q^* irresolute.

Theorem 3.3: Every $\tau_1 \tau_2$ - Q^* homeomorphism is a $\tau_1 \tau_2$ - Q^* homeomorphism.

Proof: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $\tau_1 \tau_2$ - Q^* homeomorphism.

Then f is bijective, $\tau_1 \tau_2$ - Q^* irresolute and f^{-1} is $\tau_1 \tau_2$ - Q^* irresolute.

Since every $\tau_1\tau_2 - Q^*$ irresolute is $\tau_1\tau_2 - Q^*$ continuous, f and f^{-1} are $\tau_1\tau_2 - Q^*$ continuous and so f is a $\tau_1\tau_2 - Q^*$ homeomorphism.

Remark 3.3: The following example shows that the converse of the above theorem need not be true.

Example 3.7: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}\}$. Clearly $\{b, c\}$ is $\tau_1\tau_2 - Q^*$ closed in X . Let $\sigma_1 = \{\phi, Y, \{a\}\}$ and $\sigma_2 = \{\phi, Y\}$. Then $\sigma_1\sigma_2$ - open sets on Y are $\phi, Y, \{a\}$ and $\sigma_1\sigma_2$ - closed sets on X are $\phi, Y, \{b, c\}$. Since $\{b, c\}$ is $\sigma_1\sigma_2 - Q^*$ closed in Y but $f^{-1}(\{b, c\}) = \{b, c\}$ is not $\tau_1\tau_2 - Q^*$ open in X and so f is not $\tau_1\tau_2 - Q^*$ irresolute.

Remark 3.4: The above example shows that the concepts of $\tau_1\tau_2$ - homeomorphisms and $\tau_1\tau_2 - Q^*$ homeomorphism are independent.

Definition 3.12: A bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)^*$ - irreducible if X is not empty and whenever $X = A_1 \cup A_2$ with $(\tau_1, \tau_2)^*$ - closed subsets $A_i \in X$ ($i = 1, 2$) then we have $X = A_1$ or A_2 .

Example: Let $X = \{1, 2, 3\}$ and $\tau_1 = \{\phi, X, \{1\}, \{1, 2\}\}$ and $\tau_2 = \{\phi, X, \{1\}\}$. Then $(\tau_1, \tau_2)^*$ - closed sets are $\phi, X, \{2, 3\}, \{3\}$. Then X is $(\tau_1, \tau_2)^*$ - irreducible.

Theorem: A bitopological space X is $(\tau_1, \tau_2)^*$ - irreducible if and only if every nonempty open set is $(\tau_1, \tau_2)^* - Q^*$ open.

Proof: Let X be a $(\tau_1, \tau_2)^*$ - irreducible.

Let U be any nonempty open set.

If $U = X$ then nothing to prove.

Let $U \neq X$.

Then $(\tau_1, \tau_2)^* - cl(U) \neq X$.

Then there exists an $(\tau_1, \tau_2)^* -$ open set V such that $U \cap V = \phi$.

This implies $U^c \cap V^c = X$, where U^c and V^c are proper $(\tau_1, \tau_2)^* -$ closed sets which is a contradiction to the fact that X is $(\tau_1, \tau_2)^* -$ irreducible.

Conversely assume that every $(\tau_1, \tau_2)^* -$ open set is $(\tau_1, \tau_2)^* - Q^*$ open.

We claim that X is $(\tau_1, \tau_2)^* -$ irreducible.

Then $X = A \cup B$, where A and B are proper nonempty $(\tau_1, \tau_2)^* -$ closed sets.

$A^c \cap B^c = \phi$.

Then A^c is not dense.

Then A^c is an $(\tau_1, \tau_2)^* -$ open set but not $(\tau_1, \tau_2)^* - Q^*$ open.

Hence X is irreducible.

Definition 3.13: A bitopological space (X, τ_1, τ_2) is called $\tau_i\tau_j$ - irreducible if X is not empty and whenever $X = A_1 \cup A_2$ with τ_i - closed subset $A_1 \in X$ and τ_j - closed subset $A_2 \in X$ ($i, j = 1, 2$) then we have $X = A_1$ or A_2 .

Example: Let $X = \{1, 2, 3\}$ and $\tau_1 = \{\phi, X, \{1\}, \{1, 3\}\}$ and $\tau_2 = \{\phi, X, \{2\}, \{2, 3\}\}$. Then τ_1 - closed sets are $\phi, X, \{2, 3\}, \{2\}$ and τ_2 - closed sets are $\phi, X, \{1, 3\}, \{1\}$. Then X is $\tau_i\tau_j$ -irreducible.

Theorem: A bitopological space X is $\tau_i\tau_j$ - irreducible if and only if every nonempty open set is $\tau_i\tau_j - Q^*$ open.

Proof: Let X be a $\tau_i\tau_j$ - irreducible.

Let U be any nonempty τ_j - open set.

If $U = X$ then nothing to prove.

Let $U \neq X$.

Then τ_i - $\text{cl}(U) \neq X$.

Then there exists an τ_i - open set V such that $U \cap V = \phi$.

This implies $U^c \cap V^c = X$, where U^c and V^c are proper τ_i - closed set and τ_j -closed set which is a contradiction to the fact that X is $\tau_i \tau_j$ - irreducible.

Conversely assume that every $\tau_i \tau_j$ - open set is $\tau_i \tau_j$ - Q^* open.

We claim that X is $\tau_i \tau_j$ - irreducible.

Then $X = A \cup B$, where A and B are proper nonempty τ_i - closed set and τ_j - closed set.

$$A^c \cap B^c = \phi.$$

Then A^c is not dense.

Then A^c is an $\tau_i \tau_j$ - open set but not $\tau_i \tau_j$ - Q^* open.

Hence X is irreducible.

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