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A FIXED POINT THEOREM FOR SELF MAPPING IN BANACH SPACE

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ABSTRACT

Let X be a Compact subspace of a Banach space and let f be a self mapping of X. We introduce condition for self mapping f such that f has a unique fixed point in X. In the other words, we established fixed point theorem with help of self mapping which satisfying contractive type of condition.

Mathematics Subject Classification: *47H10, 54H25.*

Keyword: *Fixed Point Theorem, Banach Space, Self Mapping, Contraction Mapping.*

1. INTRODUCTION

In recent years, nonlinear analysis have attracted much attention .The study of non contraction mapping concerning the existence of fixed points draw attention of various authors in non linear analysis. Fixed point theorem is very important in the solution of differential equations. The most famous of fixed point theorem is Brouwer's fixed point theorem. Also, a large variety of the problems of analysis and applied mathematics reduce to finding solutions of non-linear functional equations which can be formulated in terms of finding the fixed points of a non-linear mapping.(see [1,6,7,8])

Definition1.1: Let X be a metric space equipped with a distance d. A map f: X → X is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $d(f(x_1), f(x_2)) \leq \lambda\, d(x_1, x_2)$, $\forall\, x_1, x_2 \in X$.

The smallest λ for which the above inequality holds is the Lipschitz constant of f. If $\lambda \leq 1$ f is said to be non-expansive, if $\lambda < 1$ f is said to be a contraction. (See [3, 5]).

Notice: First we are giving some fundamental results:

Theorem [**Banach**] **1.2:** Let f be a contraction on a Banach space X, then f has a unique fixed point. In the other words, let (X, d) be a complete metric space and f: X → X a contraction, i. e,
$\sup \frac{d(f(x),f(y))}{d(x,y)} < 1$, then f has a unique fixed point.([2,11])

Theorem1.3: Kannan in [9] proved that "If f is self mapping of a complete metric space X into itself satisfying:

$d(f(x), f(y)) \leq \alpha[\, d(f(x), x) + d(f(y), y)]$, for all $x, y \in X$ and $\alpha \in [0, \frac{1}{2}]$, then f has a unique fixed point in X".

Theorem1.4: Fisher in [4] proved the result with $d(f(x), f(y)) \leq \alpha[\, d(f(x), x) + d(f(y), y)] + \beta d(x, y)$ for all $x, y \in X$ and $\alpha, \beta \in [0, \frac{1}{2}]$, then f has a unique fixed point in X.

Theorem1.5: A similar conclusion was also obtained by Chaterjee [10]:

$d(f(x), f(y)) \leq \alpha[\, d(f(x), y) + d(f(x), y)]$, for all $x, y \in X$ and $\alpha \in [0, \frac{1}{2}]$, then f has a unique fixed point in X.

2. A FIXED POINT THEOREM FOR SELF MAPPING IN COMPLETE METRIC SPACE

Theorem2.1: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition:

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d(f(x), f(y)) $\leq \alpha$[d(f(x), x) + d(f(y), y) - $d(x, y)$] for all x, yϵX and $\alpha \in [\frac{1}{4}, \frac{1}{2}]$, then f has a unique fixed point in X, where g is self mapping in X such that g(x)= f(2x)-x .

Proof: d(g(x)-g(y))= $\|g(x) - g(y)\|$

$= d(f(2x)-x, f(2y)-y)$

$= \|f(2x) - f(2y) - x + y\|$

$\leq \|f(2x) - f(2y)\| + \|x - y\|$

$\leq \alpha[\,\|f(2x) - 2x\| + \|f(2y) - 2y\| - d(2x, 2y)] + \|x - y\|$

$= \alpha[\,\|f(2x) - 2x\| + \|f(2y) - 2y\|] + (1-2\alpha)\|x - y\|$ (Let $1-2\alpha = \beta \rightarrow 0 \leq \beta \leq \frac{1}{2}$)

$= \alpha[\,\|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

$\rightarrow \|g(x) - g(y)\| \leq \alpha[\,\|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

With using by theorem1.4.g has a unique fixed point c in X. This means g(c) =c. Therefore, f(2c)-c=c. Hence, f(2c)=2c.

Let k=2c, then f(k)=k . Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

Theorem2.2: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition:
d(f(x), f(y)) $\leq \alpha$[d(f(x), x) + d(f(y), y) - $\frac{3}{2}d(x, y)$] for all x, yϵX and $\alpha \in [\frac{1}{6}, \frac{1}{3}]$, then f has a unique fixed point in X, where g is self mapping in X such that g(x)= f(2x)-x .

Proof: d(g(x)-g(y))= $\|g(x) - g(y)\|$

$= d(f(2x)-x, f(2y)-y)$

$= \|f(2x) - f(2y) - x + y\|$

$\leq \|f(2x) - f(2y)\| + \|x - y\|$

$\leq \alpha[\,\|f(2x) - 2x\| + \|f(2y) - 2y\| - \frac{3}{2}\, d(2x, 2y)] + \|x - y\|$

$= \alpha[\,\|f(2x) - 2x\| + \|f(2y) - 2y\|] + (1-3\alpha)\|x - y\|$ (Let $1-2\alpha = \beta \rightarrow 0 \leq \beta \leq \frac{1}{2}$)

$= \alpha[\,\|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

$\rightarrow \|g(x) - g(y)\| \leq \alpha[\,\|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

With using by theorem1.4.g has a unique fixed point c in X. This means g(c) =c. Therefore, f(2c)-c=c. Hence, f(2c)=2c.

Let k = 2c, then f(k) = k. Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

Theorem2.3: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition:

d(f(x), f(y)) $\leq \alpha$[d(f(x), x) + d(f(y), y) - $2d(x, y)$] for all x, yϵX and $\alpha \in [\frac{1}{8}, \frac{1}{4}]$, then f has a unique fixed point in X, where g is self mapping in X such that g(x)= f(2x)-x .

Proof: d(g(x)-g(y))= $\|g(x) - g(y)\|$

$= d(f(2x)-x, f(2y)-y)$

$= \|f(2x) - f(2y) - x + y\|$

$\leq \|f(2x) - f(2y)\| + \|x - y\|$

$\leq \alpha[\ \|f(2x) - 2x\| + \|f(2y) - 2y\|$ - 2 $d(2x, 2y)] + \|x - y\|$

$= \alpha[\ \|f(2x) - 2x\| + \|f(2y) - 2y\|] + (1\text{-}4\alpha)\|x - y\|$ (Let 1-2$\alpha = \beta \rightarrow 0 \leq \beta \leq \frac{1}{2}$)

$= \alpha[\ \|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

$\rightarrow\ \|g(x) - g(y)\| \leq\ \alpha[\ \|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

With using by theorem1.4.g has a unique fixed pointc in X. This means g(c) = c. Therefore, f(2c)-c=c. Hence, f(2c)=2c.

Let k=2c, then f(k)=k . Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

Theorem2.4: Let X be a complete metric space and let f be a self mapping of X into itself which satisfies the following condition:

$d(f(x), f(y)) \leq \alpha[\ d(f(x), x) + d(f(y), y) - \frac{k}{2} d(x, y)]$ for all $x, y \epsilon X$ and $\alpha \in [\frac{1}{2k}, \frac{1}{k}]$ such that $k \in \{3,4,5,6, \ldots\}$, then f has a unique fixed point in X, where g is self mapping in X such that g(x)= f(2x)-x .

Proof: $d(g(x)\text{-}g(y)) = \|g(x) - g(y)\|$

$= d(f(2x)\text{-}x, f(2y)\text{-}y)$

$= \|f(2x) - f(2y) - x + y\|$

$\leq \|f(2x) - f(2y)\| + \|x - y\|$

$\leq \alpha[\ \|f(2x) - 2x\| + \|f(2y) - 2y\| - \frac{k}{2}\ d(2x, 2y)] + \|x - y\|$

$= \alpha[\ \|f(2x) - 2x\| + \|f(2y) - 2y\|] + (1\text{-}k\alpha)\|x - y\|$ (Let 1-k$\alpha = \beta \rightarrow 0 \leq \beta \leq \frac{1}{2}$)

$= \alpha[\ \|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

$\rightarrow\ \|g(x) - g(y)\| \leq\ \alpha[\ \|g(x) - x\| + \|g(y) - y\|] + \beta\|x - y\|$.

With using by theorem1.4. g has a unique fixed point c in X. This means g(c) = c. Therefore, f(2c)-c=c. Hence, f(2c)=2c .

Let k = 2c, then f(k) = k . Therefore, f has a unique fixed point in X. The proof of theorem in this case is complete.

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