

PROPERTY OF HADAMARD PRODUCT OF MATRICES OVER SKEW FIELD

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ABSTRACT

In this paper, we give Hadamard product over skew field and relevant property by extending the property of Hadamard product of two matrices on complex domain to skew field. Especially, this paper using the method of block matrices gets a few Hadamard products' preference ordering Inequalities about Positive Semi-definite matrices $M - P$ inverse.

Keywords: Hadamard product; positive semi-definite self-conjugate matrix; skew field.

1. INTRODUCTION

Hadamard product and its property over skew field have been studied in some literatures. But we have not detailed reported for property of the Hadamard product of matrices over skew field so far. In this paper, we further study the related property and inequalities of the Hadamard product over skew field by extending the Hadamard product of two matrices on general complex domain to skew field. And based on the method of block matrices, we can obtain several Hadamard product inequalities about positive semi-definite matrices. And there are the similar conclusions on the skew field.

2. PREPARATION KNOWLEDGE

In this paper, K denotes a skew field, $K^{m \times n}$ represents the set of $m \times n$ order matrix, $M_n(K)$ represents the set of $n \times n$ order matrix over K , $GL_n(K)$ means the all of the inverse matrix of n order, A^* means the conjugate transpose matrix of A . $H(n, *)$ denotes the set of self-conjugate matrix of order n . $Z(K)$ is the centre of K , let $F = Z(K)$, so $M_n(F)$ denotes the set of communicate matrix of order n over K . Moreover, If

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \text{ then } \text{diag}A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

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Definition 1^[2]: Let $A = (a_{ij}) \in K^{m \times n}$, $B = (b_{ij}) \in K^{m \times n}$, definite the *Hadamard* product of A and B as follows:

$$A \circ B = [a_{ij}b_{ij}] = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}b_{m1} & a_{m2}b_{m2} & \cdots & a_{mn}b_{mn} \end{pmatrix}.$$

Definition 2^[1]: Let R is a non-commutative principal ideal domian with involutorial anti-automorphism σ , $A \in R^{m \times n}$, if there exits $X \in R^{n \times m}$ such that:

$$\begin{aligned} AXA &= A, \\ XAX &= X, \\ (AX)^* &= AX, \\ (XA)^* &= XA. \end{aligned}$$

Then, we say X is a *Moore – Penrose* inverse of A , denoted A^+ .

Definition 3^[1]: Let $A \in H(n,*)$, $H(n,*)$ denotes the set of self-conjugate matrix of order n . If $\alpha^* A \alpha \geq 0$ holds for any n dimension and non-zero column vector, then we say A is a positive semi-definite self-conjugate matrix of order n . we rewrite it: $A \geq 0$

Definition 4^[1]: Let $A \in M_n(R)$, if there is a $B \in M_n(R)$, such that $AB = BA = I_n$, then we call A is a inverse matrix of n order. The inverse of A is denoted A^{-1} .

The following lemmas are the basis of the conclusions we talking:

Lemma 1^{[3][4]}: $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in M_n(K)$ and $M \geq 0$, where $A, B, C \in M_n(K)$, then $C - B^* A^+ B \geq 0$

Lemma 2^[4]: For any $A, B \in K^{m \times n}$, we have $A \circ B = M^T (A \otimes B) M$, where M is the $n^2 \times n$ selection matrix, N is the $m^2 \times m$ selection matrix. (M and N are the matrices which elements only are 1 and 0, and $M^T M = I_n, N^T N = I_m$), $A \otimes B$ is the *Kronecker* product of A and B . This lemma is the connection of the *Kronecker* and *Hadamard* product.

Lemma 3: Let $A, B \in M_n(K)$, and $A \geq 0, B \geq 0$, then $M_i \geq 0 (i = 1, 2, 3, 4)$,

$$\text{where } M_1 = \begin{bmatrix} BB^+ & B \\ B & B^2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A^+ & AA^+ \\ A^+A & A \end{bmatrix},$$

$$M_3 = \begin{bmatrix} B^2 & B \\ B & BB^+ \end{bmatrix}, \quad M_4 = \begin{bmatrix} A & AA^+ \\ AA^+ & A^+ \end{bmatrix}.$$

Proof: $\begin{bmatrix} I_n & 0 \\ -B & I_n \end{bmatrix} M_1 \begin{bmatrix} I_n & -B \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} BB^+ & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$

$$\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix} M_2 \begin{bmatrix} I_n & -A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A^+ & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

Similarly, $M_3 \geq 0, M_4 \geq 0.$

Lemma 4: Let $A, B \in M_n(K)$, moreover $A \geq 0, B \geq 0$, if $\begin{pmatrix} A & B \\ B & A \end{pmatrix} \geq 0$, then $A \geq B$.

Proof: $2(A - B) = (A - B, B - A) \begin{bmatrix} I_n \\ -I_n \end{bmatrix} = (-I_n, I_n) \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} I_n \\ -I_n \end{bmatrix} \geq 0,$

So $A - B \geq 0$, 即 $A \geq B$.

3. PROPERTY AND THEOREM

Hadamard meet the following basic properties:

(1) $A = (a_{ij}) \in K^{m \times n}$, null matrix $\mathbf{0} \in M_n(K)$, 则 $A \circ \mathbf{0} = \mathbf{0} \circ A = \mathbf{0}$.

(2) $(A \circ B)^T = A^T \circ B^T$.

(3) $c(A \circ B) = (cA) \circ B$, c is a real number.

(4) The *Hadamard* product of $A = (a_{ij}) \in K^{m \times m}$ and I_m is a $m \times m$ diagonal matrix namely:

$$A \circ I_m = I_m \circ A = \text{diag}(A) = \text{diag}(a_{11}, a_{22} \cdots a_{mm}).$$

(5) $A \circ (B \circ C) = (A \circ B) \circ C = A \circ B \circ C$

$$(A \pm B) \circ C = A \circ C \pm B \circ C$$

$$(A + B) \circ (C + D) = A \circ C + A \circ D + B \circ C + B \circ D.$$

(6) If $A = (a_{ij}) \in K^{m \times m}, C = (c_{ij}) \in K^{m \times m}$

$$B = (b_{ij}) \in K^{n \times n}, D = (d_{ij}) \in K^{n \times n}$$

Then $(A \oplus B) \circ (C \oplus D) = (A \circ C) \oplus (B \circ D)$.

Proof: $A \oplus B = \begin{bmatrix} A & 0_{m \times n} \\ 0_{n \times m} & B \end{bmatrix}, C \oplus D = \begin{bmatrix} C & 0_{m \times n} \\ 0_{n \times m} & D \end{bmatrix}$

$$(A \oplus B) \circ (C \oplus D) = \begin{bmatrix} A \circ C & 0_{m \times n} \\ 0_{n \times m} & B \circ D \end{bmatrix}$$

$$(A \circ C) \oplus (B \circ D) = \begin{bmatrix} A \circ C & 0_{m \times n} \\ 0_{n \times m} & B \circ D \end{bmatrix}$$

This proposition is proved.

(7) If A, B, D are the square matrix of $M_n(K)$, D is a diagonal matrix, so $(DA) \circ (BD) = D(A \circ B)D$.

(8) $tr(A \circ B) \neq tr(A)tr(B)$.

(9) Because the skew field (decision ring) is non-commutative, $A \circ B \neq B \circ A$.

For example: $A = \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, B = \begin{pmatrix} j & k \\ i & 1 \end{pmatrix},$

Then $A \circ B = \begin{pmatrix} j & ik \\ ji & k \end{pmatrix} = \begin{pmatrix} j & -j \\ -k & k \end{pmatrix}, B \circ A = \begin{pmatrix} j & ki \\ ij & k \end{pmatrix} = \begin{pmatrix} j & j \\ k & k \end{pmatrix},$

So $A \circ B \neq B \circ A$.

(10) $A, B \in K^{m \times n}$, If A, B are positive semi-definite, then its *Hadamard* product $A \circ B$ is also positive semi-definite.

Theorem 1: If $A, B, C \in K^{m \times n}$, then $tr[A^T(B \circ C)] = tr[(A^T \circ B^T)C]$.

Proof: Noted the diagonal elements of $A^T(B \circ C)$ and $(A^T \circ B^T)C$ are common, that is :

$$[A^T(B \circ C)]_{ii} = \sum_{k=1}^n a_{ki} b_{ki} c_{ki} = [(A^T \circ B^T)C]_{ii}.$$

Theorem 2: If $A, B \in K^{m \times n}$, then $rank(A \circ B) \leq (rankA)(rankB)$.

Proof: Any matrix with rank r can be written as combination of r matrices with rank 1. Every matrix with rank 1 is the

outer product of two vectors. If $rankA = r_1, rankB = r_2$, then $A = \sum_{i=1}^{r_1} x_i y_i^T, B = \sum_{j=1}^{r_2} u_j v_j^T$, where x_i, u_j and

y_i, v_j are the linear independent column vector over

$K, i = 1, 2, \dots, r_1, j = 1, 2, \dots, r_2$. Therefore, $A \circ B = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} (x_i \circ u_j)(y_i \circ v_j)^T$. Then

$$\begin{aligned} \text{rank}(A \circ B) &= \text{rank} \left[\sum_{i=1}^{r_1} \sum_{j=1}^{r_2} (x_i \circ u_j)(y_i \circ v_j)^T \right] \leq \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} \text{rank} \left[(x_i \circ u_j)(y_i \circ v_j)^T \right] \\ &\leq \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} 1 = r_1 r_2 = (\text{rank}A)(\text{rank}B). \end{aligned}$$

This suggests that $A \circ B$ could be written as combination of at most $r_1 r_2$ matrices with 1 rank.

So, $\text{rank}(A \circ B) \leq (\text{rank}A)(\text{rank}B)$.

Another method: Because $A \circ B = M^T (A \otimes B) N$, according to the property of matrix of rank:

$$\text{rank}(A \circ B) = \text{rank}[M^T (A \otimes B) N] \leq \text{rank}(A \otimes B).$$

Next we should show $\text{rank}(A \otimes B) = (\text{rank}A)(\text{rank}B)$

Let $r(A) = s$, then there exists $P \in GL_m(K), Q \in GL_n(K)$ such that $PAQ = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$. For $B \in K^{m \times n}$, let

$r(B) = t$ there exists $S \in GL_m(K), T \in GL_n(K)$ such that

$$\begin{aligned} SBT &= \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix}. \text{because } (P \otimes S)(A \otimes B)(P \otimes S) = (PAQ) \otimes (SBT) = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} = \\ &\begin{pmatrix} I_{st} & 0 \\ 0 & 0 \end{pmatrix}. \text{Because } P \otimes S, Q \otimes T \text{ are invertible matrix, } r(A \otimes B) = r \begin{pmatrix} I_{st} & 0 \\ 0 & 0 \end{pmatrix} = st = r(A) r(B). \end{aligned}$$

So, $\text{rank}(A \circ B) \leq (\text{rank}A)(\text{rank}B)$.

Theorem 3: Let $A, B \in H(n, *)$ and $A \geq 0, B \geq 0$, then $A^+ \circ B^+ \geq [(A^+A) \circ (B^+B)](A \circ B)^+ [(AA^+) \circ (BB^+)]$.

Proof: By lemma3 and theorem 1:

$$\begin{pmatrix} A \circ B & (AA^+) \circ (BB^+) \\ (A^+A) \circ (B^+B) & A^+ \circ B^+ \end{pmatrix} = \begin{pmatrix} A & AA^+ \\ A^+A & A^+ \end{pmatrix} \circ \begin{pmatrix} B & BB^+ \\ B^+B & B^+ \end{pmatrix} \geq 0.$$

According to the definition of Moore – Penrose inverse:

$$[(AA^+) \circ (BB^+)]^* = (AA^+)^* \circ (BB^+)^* = (A^+A) \circ (B^+B)$$

Remark: $A \circ B = C, (AA^+) \circ (BB^+) = D, A^+ \circ B^+ = G$

Then $\begin{pmatrix} A \circ B & (AA^+) \circ (BB^+) \\ (A^+A) \circ (B^+B) & A^+ \circ B^+ \end{pmatrix} = \begin{pmatrix} C & D \\ D^* & G \end{pmatrix} \geq 0$

According to lemma 2, $G - D^*C^+D \geq 0$,

so, $A^+ \circ B^+ - [(A^+A) \circ (B^+B)](A \circ B)^+ [(AA^+) \circ (BB^+)] \geq 0$.

Corollary 1: Assume that $A, B \in H(n, *)$ and $A \geq 0, B > 0$, then

$$A^+ \circ B^- \geq (\text{diag} A^+ A)(A \circ B)^+ (\text{diag} AA^+)$$

Proof: According to theorem 1, let $B^+ = B^-$, then $B^- B = BB^- = I$, on the basis of property (4), We can come to this conclusion easily.

Corollary 2: Assume that $A, B \in H(n, *)$ and $A > 0, B > 0$, then $A^- \circ B^- \geq (A \circ B)^-$.

Proof: Let $A^+ = A^-, B^+ = B^-$, the result can be obtained.

Theorem 4: Assume that $A, B \in H(n, *)$ and $A \geq 0, B \geq 0$ then $A^2 \circ B^2 \geq (A \circ B)[(AA^+) \circ (BB^+)]^+(A \circ B)$

Proof: Combing the lemma 3 and theorem 1

$$\begin{pmatrix} (AA^+) \circ (BB^+) & A \circ B \\ A \circ B & A^2 \circ B^2 \end{pmatrix} = \begin{pmatrix} AA^+ & A \\ A & A^2 \end{pmatrix} \circ \begin{pmatrix} BB^+ & B \\ B & B^2 \end{pmatrix} \geq 0$$

By lemma 2, $A^2 \circ B^2 - (A \circ B)[(AA^+) \circ (BB^+)]^+(A \circ B) \geq 0$.

Corollary 3: Assume that $A, B \in H(n, *)$ and, then $A^2 \circ B^2 \geq (A \circ B)^2$.

Theorem 5: A is a self-conjugate matrix on $M_n(F)$, and $A \geq 0$, then

$$(AA^+) \circ A^2 \geq A \circ A.$$

Proof: According to lemma 3 and theorem 1:

$$\begin{pmatrix} A^2 \circ (AA^+) & A \circ A \\ A \circ A & A^2 \circ (AA^+) \end{pmatrix} = \begin{pmatrix} (AA^+) \circ A^2 & A \circ A \\ A \circ A & A^2 \circ (AA^+) \end{pmatrix} = \begin{pmatrix} AA^+ & A \\ A & A^2 \end{pmatrix} \circ \begin{pmatrix} A^2 & A \\ A & AA^+ \end{pmatrix} \geq 0$$

By lemma 4, $(AA^+) \circ A^2 \geq A \circ A$.

Corollary 4: A is a self-conjugate matrix on $M_n(F)$, then $\text{diag} A^2 \geq A \circ A$

Corollary 5: By corollary 4, if all the diagonal elements of A^2 are 1, then $A \circ A \leq I_n$

Theorem 6: A is a self-conjugate matrix on $M_n(F)$, and $A \geq 0$, then $A \circ A^+ \geq (A^+ A) \circ (AA^+)$

Proof: Combing the lemma 3 and theorem 1:

$$\begin{pmatrix} A \circ A^+ & (AA^+) \circ (A^+ A) \\ (A^+ A) \circ (AA^+) & A \circ A^+ \end{pmatrix} = \begin{pmatrix} A \circ A^+ & (AA^+) \circ (AA^+) \\ (A^+ A) \circ (A^+ A) & A^+ \circ A \end{pmatrix} = \begin{pmatrix} A & AA^+ \\ A^+ A & A^+ \end{pmatrix} \circ \begin{pmatrix} A^+ & AA^+ \\ A^+ A & A \end{pmatrix} \geq 0$$

by lemma 4, $(AA^+) \circ A^2 \geq A \circ A$

Theorem 7: Assume that $A, B, M \in K^{m \times n}$, $A \geq 0, B \geq 0, M \geq 0$, for any real number a, b , we have:

$$(a^2 + b^2)(A^* A \circ B^* B) + 2ab(A^* B \circ B^* A) \geq (a + b)^2(A^* M \circ B^* M)(M^* M \circ MM^*)^+(M^* A \circ M^* B)$$

Proof: Remark: $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}$, where $L_1 = M \otimes M$, $L_2 = aA \otimes B + bB \otimes A$.

$$S = L^*L = \begin{pmatrix} L_1^* \\ L_2^* \end{pmatrix} \begin{pmatrix} L_1 & L_2 \end{pmatrix} = \begin{pmatrix} L_1^*L_1 & L_1^*L_2 \\ L_2^*L_1 & L_2^*L_2 \end{pmatrix} \geq 0$$

Though calculation:

$$L_1^* L_1 = (M^*M \otimes M^*M)$$

$$L_1^* L_2 = a M^* A \otimes M^*B + bM^*B \otimes M^*A$$

$$L_2^*L_1 = aA^*M \otimes B^*M + bB^*M \otimes A^*M$$

$$L_2^*L_2 = a^2(A^*A \otimes B^*B) + b^2(B^*B \otimes A^*A) + ab[(A^*B \otimes B^*A) + (B^*A \otimes A^*B)]$$

Combing the lemma 3 and theorem 1, the result can be held up.

Corollary 6: If $A, B \in K^{m \times n}$, and $k \in [-1, 1]$, then $AA^* \circ BB^* + kAB^* \circ BA^* \geq (1+k)(A \circ B)(A \circ B)^*$

Proof: By theorem8, let $a^2 + b^2 = 1$, $2ab = k$, $M = I$, we can see the results easily.

Corollary 7: If $A, B \in K^{m \times n}$, then $A^*A \circ B^*B \geq (A^* \circ B^*)(A \circ B)$

Proof: By theorem8, let $a = 1, b = 0, M = I$, the results can be held.

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