



# A CONFORMAL VERTEX ALGEBRA STRUCTURE ON THE REPRESENTATION $V_Q$ OF THE AFFINE LIE ALGEBRA $D_n^{(1)}$

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## ABSTRACT

The vertex operator structure on the representation  $V_Q$  of untwisted affine algebra associated with  $D_n^{(1)}$  is studied by using the representation theory of Lie algebra. Moreover, it is proved that  $V_Q$  is a vertex operator algebra according to calculus methods of formal distributions, and then the conformal vector on the vertex algebra  $V_Q$  is given.

**Key words:** vertex algebra; vertex operator;  $n$ -th product.

## 1. INTRODUCTION

The physicists brought forward the concept of vertex operation algebra in studying the theory of field and string. It is important in studying representation theory and finite group. Meurmen and Lepowsky solved the Guss of McKay-Thompson with this theory. And Borchers used the vertex operation algebra and Kac-Moody Lie algebra to solve the famous problem of the Monstrous Moonshine Conjecture and won fields award in 1998. Frenkel and Kac<sup>[1,2]</sup> had constructed the level-one representations of affine Kac-Moody algebras  $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$  by means of vertex operators in 1981. In addition, Xu and Jiang<sup>[3]</sup> have introduced another set of vertex operators in 1990 which are constructed for the level-one representations of the cases  $B_n^{(1)}$  and  $G_2^{(1)}$ . Xu<sup>[4]</sup> gave the level-one representations of the affine Lie algebras with first kind in 1991.

Using these vertex operators to construct a vertex operator algebra has been an important subject of study. The representations  $V_Q$  of Kac-Moody Lie algebra associated to ..are constructed, which are based on a certain untwisted or twisted vertex operators, and it is proofed to be a vertex operator algebra in Ref.[5]. In this paper, we use the vertex operators of affine Lie algebra  $D_n^{(1)}$  with first kind to construct vertex operator algebras. But Jacobi Identity for the definition of vertex operator algebra is very complicated. We instead it with the axiom of locality to construct vertex operator algebra.

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## 2. UNTWISTED VERTEX REPRESENTATION $V_Q$ OF $D_n^{(1)}$

In this section, we briefly introduce the structure of  $V_Q$  and vertex operators  $Y(v, z)$  on  $V_Q$ . Let  $B(x, y)$  be the Killing form of a finite dimensional complex simple Lie algebra  $SO(2n, \mathbb{C})$ . Let  $\mathfrak{H}$  ( $\dim(\mathfrak{H}) = n$ ) be a Cartan subalgebra of  $SO(2n, \mathbb{C})$  and  $\Delta$  be the root system. Then  $SO(2n, \mathbb{C}) = \mathfrak{H} + \sum_{\alpha \in \Delta} g_\alpha$ ,  $g_\alpha$  is the root subspace decomposition of by the Cartan subalgebra  $\mathfrak{H}$ . Denote  $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{H}^*$  a simple root system, where  $\mathfrak{H}^*$  is the dual space of  $\mathfrak{H}$ . The root lattice

$$L = \left\{ \alpha = \sum_{i=1}^n m_i \alpha_i \mid m_1, \dots, m_n \in \mathbb{Z} \right\} \subset \mathfrak{H}_R^* \quad (1)$$

is an abelian addition group in the real linear space  $\mathfrak{H}_R^*$ . Then  $\mathfrak{H}_R^*$  has an inner product  $(x, y) = c_0 B(x, y)$ ,  $\forall x, y \in \mathfrak{H}_R^*$ ,

where  $c_0$  is a positive constant. The group algebra  $\mathbb{C}(e^L)$  of  $L$  is an abelian associative algebra with the basis

$$\{e^\alpha \mid \alpha \in L\}, \text{ where } e^0 = 1 \text{ and } e^\alpha e^\beta = e^{\alpha+\beta}.$$

Denote  $h_i(m) = t^m \otimes \alpha_i$ ,  $m \in \mathbb{Z}$ ,  $1 \leq i \leq n$ ,  $X_m(\alpha) = t^m \otimes e_\alpha$ ,  $\forall \alpha \in \Delta$ . where  $t$  is a complex parameter.

Let  $S^-$  be the complex linear space spanned by the basis  $1, h_i(-m)$ ,  $m \in \mathbb{Z}^+$ ,  $1 \leq i \leq n$ . Denote  $S(S^-)$  be the symmetric tensor algebra over  $\mathbb{C}$  generated by  $S^-$  with the product  $\vee$ . Then  $S(S^-)$  is a commutative associated algebra with the unit element 1 and has a basis

$$1, h_{i_1}(-m_1) \vee \dots \vee h_{i_s}(-m_s), 1 \leq i_1 \leq \dots \leq i_s \leq n, m_1, \dots, m_s \in \mathbb{Z}^+, s \in \mathbb{Z}^+$$

Let  $V = S(S^-) \otimes \mathbb{C}(e^L)$ . The formal linear combination of finite or infinite elements of the basis forms a complete space  $V_Q$  of  $V$ . It is well known that  $V_Q$  is an associative algebra with

$$(h_{i_1}(-m_1) \vee \dots \vee h_{i_s}(-m_s)) \otimes e^\alpha = (1 \otimes e^\alpha) \prod_{k=1}^s (h_{i_k}(-m_k) \otimes e^0). \quad (2)$$

Hence the representation space  $V$  has a basis

$$1 \otimes e^\beta, (h_{i_1}(-m_1) \vee \dots \vee h_{i_s}(-m_s)) \otimes e^\beta = (1 \otimes e^\beta) \prod_{k=1}^s ((h_{i_k}(-m_k) \otimes e^0),$$

where  $\beta \in L$ ,  $1 \leq i_1 \leq \dots \leq i_s \leq n$ ,  $m_1, \dots, m_s \in \mathbb{Z}^+$ ,  $s = 1, 2, \dots$ .

For any  $u \otimes e^\beta \in V_Q$ , the degree of  $u \otimes e^\beta$  is defined by

$$\deg(u \otimes e^\beta) = \deg(u) + \frac{1}{2}(\beta, \beta),$$

where  $\deg(u)$  is defined by

$$\deg(1) = 0, \deg[h_{i_1}(-m_1) \vee \dots \vee h_{i_s}(-m_s)] = \sum_{k=1}^s m_k.$$

Now, we introduce some linear operators acting on  $V$  for the definition of the vertex operator representation of the affine Lie algebra, vertex operator algebra and vertex algebra.

(I) Let  $D : V_Q \rightarrow V_Q$  be a linear operator, which is defined by

$$D(v \otimes e^\beta) = d \, g \, v \otimes e^\beta (v \otimes e^\beta). \quad (3)$$

(II) Let  $\partial_{h_i(m_i)}, 1 \leq i \leq n, m_i \in \mathbb{Z}$  be the linear differential operators acting on the linear space  $S(S^-)$ , which are

$$\text{defined by } \partial_{h_i(m_i)}(h_j(-m_j)) = m_i \delta_{m_i, -m_j}(\alpha_i, \alpha_j), m_i, m_j \in \mathbb{Z}^+. \quad (4)$$

Let  $\alpha_i(m_i)$  be a linear operator acting on  $V_Q$ , which is defined by

$$\begin{cases} \alpha_i(-m_i)(v \otimes e^\beta) = (h_i(-m_i) \vee v) \otimes e^\beta, & \text{when } m_i \in \mathbb{Z}^+, \\ \alpha_i(0)(v \otimes e^\beta) = (\alpha_i, \beta)(v \otimes e^\beta), \\ \alpha_i(m_i)(v \otimes e^\beta) = \partial_{h_i(m_i)}(v) \otimes e^\beta, & \text{when } m_i \in \mathbb{Z}^-. \end{cases}$$

where  $v \in S(S^-), e^\beta \in C(e^L)$ .

**Lemma 1:**  $[\alpha_i(m_i), \alpha_j(q_j)] = m_i \delta_{m_i, -q_j}(\alpha_i, \alpha_j) \text{id}, m_i, q_j \in \mathbb{Z}$ .

**Proof:** This formula has been easily proved in Refs.[6,7].

Let  $\alpha = \sum_{i=1}^n a_i \alpha_i \in L$ . The linear operator  $\alpha(m)$  acting on  $V$  is defined by  $\alpha(m) = \sum_{i=1}^n a_i \alpha_i(m), \forall m \in \mathbb{Z}$ .

By the induction, we have

**Lemma 2:** Let  $\alpha(m) = \sum_{i=1}^n a_i \alpha_i(m), \beta(q) = \sum_{j=1}^n b_j \alpha_j(q)$ , then

$$[\alpha(m), \beta(q)] = m \delta_{m+q, 0}(\alpha, \beta) \text{id} \quad (5)$$

Particularly,

$$\exp(\alpha(m)) \exp(\beta(-m)) = \exp(m(\alpha, \beta)) \exp(\beta(-m)) \exp(\alpha(m))$$

(III) The mapping  $\varepsilon : L \times L \rightarrow \{e^{i\theta} | 0 \leq \theta < 2\pi\}$  is called the  $\varepsilon$ -mapping, if  $\varepsilon$  satisfies the following conditions:

(i)  $\varepsilon(0, \beta) = \varepsilon(\beta, 0) = 1, \forall \beta \in L$ ;

(ii)  $\varepsilon(\alpha, \beta) = (-1)^{(\alpha, \beta)} \varepsilon(\beta, \alpha), \forall \alpha, \beta \in L$ ;

(iii)  $\varepsilon(\beta + \gamma, \alpha) \varepsilon(\beta, \gamma) = \varepsilon(\beta, \alpha + \gamma) \varepsilon(\gamma, \alpha), \forall \alpha, \beta, \gamma \in L$ .

(iv) Let  $x(\alpha, z) : \mathbb{C}(e^L) \rightarrow \mathbb{C}(e^L)$  be a linear operator, which is defined by  $x(\alpha, z)(v \otimes e^\gamma) = \varepsilon(\alpha, \gamma) z^{(\alpha, \gamma)} (v \otimes e^{\alpha+\gamma})$  (6)

Let  $X(\alpha, z) = \sum_{m=-\infty}^{\infty} X_m(\alpha) z^{-m}$  be the Laurent series of  $z$ . Then

$$X(\alpha, z)(v \otimes e^\gamma) = E^+(\alpha, z) E^-(\alpha, z) x(\alpha, z)(v \otimes e^\gamma) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \alpha(-m) \exp \sum_{m=1}^{\infty} \frac{-z^{-m}}{m} \alpha(m) \varepsilon(\alpha, \gamma) z^{(\alpha, \gamma)} (v \otimes e^{\alpha+\gamma}).$$

**Theorem 1:** The vertex operator representation  $(\rho, V)$  of affine Lie algebras with the first kind can be defined on the generators by

$$\begin{cases} \rho(c) = \text{id}, \\ \rho(d) = D, \\ \rho(t^m \otimes \alpha_i) = \alpha_i(m), \quad 1 \leq i \leq n, \quad m \in \mathbb{Z}, \\ \rho(t^m \otimes e_\alpha) = X_m(\alpha), \quad \alpha \in \Delta, \quad m \in \mathbb{Z}. \end{cases}$$

This theorem is proofed in Pre[8]. In this case, the multiplication table of the affine Lie algebra is

$$[\text{id}, D] = [\text{id}, \alpha_i(m)] = [\text{id}, X_m(\alpha)] = 0, [D, \alpha_i(m)] = m\alpha_i(m),$$

$$[D, X_m(\alpha)] = mX_m(\alpha), [\alpha_i(m), \alpha_j(k)] = m\delta_{m,-k}(\alpha_i, \alpha_j)\text{id},$$

$$[\alpha_i(m), X_k(\alpha)] = (\alpha, \alpha_i)X_{m+k}(\alpha),$$

$$[X_m(\alpha), X_k(-\alpha)] = \varepsilon(\alpha, -\alpha)(\alpha(m+k) + m\delta_{m,-k})\text{id},$$

$$[X_m(\alpha), X_k(\beta)] = 0, \alpha, \beta \in \Delta, \alpha + \beta \notin \Delta \cup \{0\}, [X_m(\alpha), X_k(\beta)] = \varepsilon(\alpha, \beta)X_{m+k}(\alpha + \beta), \alpha, \beta, \alpha + \beta \in \Delta.$$

### 3. THE STRUCTURE OF VERTEX ALGEBRA ON $V_Q$

**Definition 1:** A complex linear space  $V$  is called a vertex algebra, if there exist a set of linear operators (every linear operator is called a field) for  $v$  :

$$Y(v, z) = \sum_{m \in \mathbb{Z}} v_{(m)} z^{-m-1} \in \text{End}V[[z, z^{-1}]], \quad (7)$$

such that given any  $v, w \in V$ , there is a positive integer  $m_0 = m_0(v, w)$ , such that  $v_{(m)}(w) = 0, \forall m > m_0$ . And there is a fixed vector  $|0\rangle \in V$ , which is called by the vacuum vector, such that

(i) (vacuum)

$$Y(|0\rangle, z) = \text{id}_V, \quad Y(v, z)|0\rangle|_{z=0} = v.$$

(ii) (translation convariance)

$T \in \text{End}(V)$  is defined by  $T(v) = v_{(-2)}|0\rangle, \forall v \in V$ .  $T$  is called the infinitesimal translation operator, if  $T$  is a

derivation on  $V$  and satisfies the condition:  $\text{ad}(T) = \partial_z$  acting on any linear operator  $Y(u \otimes e^\gamma, z)$ .

(iii) (locality) There exists a positive integer  $N$ , such that

$$(z-w)^N [Y(u, z), Y(v, w)] = 0, \forall u, v \in V.$$

This definitions is same in Pre. [9, 10]

Define the map

$$Y(\cdot, z) : V_Q \longrightarrow (\text{End}V_Q)[[z, z^{-1}]]$$

$$v \longrightarrow \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

by the following way:

$$Y(1, z) = \text{Id}_{V_Q}; \quad Y(h_i(-1) \otimes 1, z) = H_i(z) = \sum_{m \in \mathbb{Z}} H_i(m) z^{-m-1};$$

$$Y(h_i(-m) \otimes 1, z) = \partial^{(m-1)} H_i(z); \quad Y(1 \otimes e^\alpha) = X(\alpha, z) = E^+(\alpha, z)E^-(\alpha, z)x(\alpha, z);$$

$$Y(h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s) \otimes 1, z) =: \partial^{(m_1-1)} H_{i_1}(z) \cdots \partial^{(m_s-1)} H_{i_s}(z); \quad Y(h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s) \otimes e^\beta, z) =: \partial^{(m_1-1)} H_{i_1}(z) \cdots \partial^{(m_s-1)} H_{i_s}(z) Y(1 \otimes e^\beta, z);$$

where  $:\cdots:$  is the normal order of fields or operators,  $\partial^{(m)} = \frac{\partial^m}{m!}$  is a differential operator, and the vacuum vector  $|0\rangle = 1 \otimes e^0$ .

In the following, we shall check that  $(V, Y(v \otimes e^\alpha, z))$  is a vertex algebra, i.e., it satisfies the three axioms of the vertex algebra.

### 3.1. The vacuum axiom

- (1)  $Y(1 \otimes 1, z) = Id_v$  ;
- (2)  $Y(h_i(-m) \otimes 1, z)1 \otimes 1|_{z=0} = h_i(-m) \otimes 1$  ;
- (3)  $Y(v \otimes e^\alpha, z)1 \otimes 1|_{z=0} = v \otimes e^\alpha$  .

This formula can be easily proved by the Theorem 1.

### 3.2. The translation covariance axiom

Let  $v = \prod_{k=1}^s h_{i_k}(-m_k)$ ,  $v_k = \prod_{1 \leq j \leq s, j \neq k} h_{i_j}(-m_j)$ ,  $m_i \in \mathbb{Z}^+$ . Then

$$T(v \otimes e^\gamma) = \gamma(-1)(v \otimes e^\gamma) + \sum_{k=1}^s m_k h_{i_k}(-(1+m_k))(v_k \otimes e^\gamma) .$$

$T$  is a derivation which acts on  $V_Q$ . Particularly,

$$T(1 \otimes e^\gamma) = \gamma(-1)(1 \otimes e^\gamma), \quad T(h_i(-m) \otimes 1) = m h_i(-m-1) \otimes 1.$$

**Lemma 3:** Given  $i = 1, 2, \dots, n$ ,  $m \in \mathbb{Z}_+$ ,  $m > 0$ ,  $\alpha \in L$ , then

- (i)  $\text{ad}(T)\alpha(-m) = m\alpha(-1+m)$  ;
  - (ii)  $\text{ad}(T)\alpha(1)(v \otimes e^{\alpha+\gamma}) = -(\alpha, \alpha + \gamma)(v \otimes e^{\alpha+\gamma})$  ;
  - (iii)  $\text{ad}(T)\alpha(m) = -m\alpha(m-1)$  ;
  - (iv)  $\text{ad}(T)x(\alpha, z) = \alpha(-1)x(\alpha, z)$  ;
  - (v)  $\text{ad}(T)E^+(\alpha, z) = E^+(\alpha, z) \sum_{m=2}^{\infty} z^{m-1} \alpha(-m)$  ;
  - (vi)  $\text{ad}(T)E^-(\alpha, z) = [\sum_{m=1}^{\infty} z^{-m-1} \alpha(m) + z^{-1}(\alpha, \alpha + \gamma)]E^-(\alpha, z)$  ;
- $$E^-(\alpha, z)[T, x(\alpha, z)](v \otimes e^\gamma) = [\alpha(-1) - z^{-1}(\alpha, \alpha)]E^-(\alpha, z)x(\alpha, z)(v \otimes e^\gamma).$$

**Proof:** The formulas (i-iii, v) is easily proved[11]. Now, we only check the formulas (iv) and (v).

$$\begin{aligned} \text{ad}(T)E^+(\alpha, z) &= \text{ad}(T) \prod_{m=1}^{\infty} \exp \frac{1}{m} z^m \alpha(-m) \\ &= \sum_{j=1}^{\infty} \prod_{m=1}^{j-1} \exp \frac{1}{m} z^m \alpha(-m) \text{ad}(T) \exp \frac{1}{j} z^j \alpha(-j) \prod_{m=j+1}^{\infty} \exp \frac{1}{m} z^m \alpha(-m) \\ &= E^+(\alpha, z) \sum_{j=1}^{\infty} z^j \alpha(-(1+j)). \end{aligned}$$

Hence (iv) holds.

**Lemma 4:** The infinitesimal translation operator  $T$  satisfies the translation covariance axioms

$$[T, Y(v \otimes e^\gamma)] = \partial_z Y(v \otimes e^\gamma) . \quad (8)$$

**Proof:** Since  $\partial_z$  is a differential operator acting on  $Y(u \otimes e^\alpha)$  about  $z$ , then

$$\begin{aligned} \partial_z (Y(1 \otimes e^\alpha, z))(v \otimes e^\gamma) &= \partial_z (E^+(\alpha, z)E^-(\alpha, z)x(\alpha, z))(v \otimes e^\gamma) \\ &= \partial_z (E^+(\alpha, z))E^-(\alpha, z)x(\alpha, z) + E^+(\alpha, z)\partial_z (E^-(\alpha, z))x(\alpha, z) + E^+(\alpha, z)E^-(\alpha, z)\partial_z (x(\alpha, z))(v \otimes e^\gamma) \\ &= E^+(\alpha, z)\left[\sum_{m=1}^{\infty} z^{m-1}\alpha(-m) + \sum_{m=1}^{\infty} z^{-m-1}\alpha(m) + z^{-1}(\alpha, \gamma)\right]E^-(\alpha, \sqrt{z})(\partial_z x(\alpha, \sqrt{z}))(v \otimes e^\gamma). \end{aligned}$$

By the formulas (iv), (v) and (vi) of Lemma 3, we have

$$\begin{aligned} [T Y(1 \otimes e^\alpha, z)](v \otimes e^\gamma) &= [T E^+(\alpha, z)E^-(\alpha, z)x(\alpha, z)](v \otimes e^\gamma) \\ &= [[T E^+(\alpha, z)]E^-(\alpha, z)x(\alpha, z) + E^+(\alpha, z)[T E^-(\alpha, z)]x(\alpha, z) + E^+(\alpha, z)E^-(\alpha, z)[T x(\alpha, z)]](v \otimes e^\gamma) \\ &= E^+(\alpha, z)\left[\sum_{m=1}^{\infty} z^{m-1}\alpha(-m) + \sum_{m=1}^{\infty} z^{-m-1}\alpha(m) + z^{-1}(\alpha, \gamma)\right]E^-(\alpha, z)x(\alpha, z)(v \otimes e^\gamma), \end{aligned}$$

Therefore

$$[T Y(1 \otimes e^\alpha, z)] = \partial_z (Y(1 \otimes e^\alpha, z)).$$

It is easy to see that

$$[T Y(h_i(-m) \otimes e^0, z)] = \partial_z (Y(h_i(-m) \otimes e^0, z))$$

by the formulas (ii), (iv) and (vi) of Lemma 3. By the induction, we have

$$[T Y(v \otimes e^0, z)] = \partial_z (Y(v \otimes e^0, z)),$$

and

$$[T Y(u \otimes e^\alpha, z)] = \partial_z (Y(u \otimes e^\alpha, z)),$$

where  $u = \prod_{k=1}^s h_i(-m_k)$ . Then

$$\begin{aligned} \text{ad}(T)Y(h_{i_s}(-m_0) \vee u \otimes e^\alpha, z) &= \text{ad}(T) : \partial^{(m_0-1)} H_{i_s}(z)Y(u \otimes e^\alpha) : \\ &= \text{ad}(T)[\partial^{(m_0-1)} H_{i_s}(z)_+ Y(u \otimes e^\alpha)] + \text{ad}(T)[Y(u \otimes e^\alpha) \partial^{(m_0-1)} H_{i_s}(z)_-] \\ &= [\text{ad}(T) \partial^{(m_0-1)} H_{i_s}(z)_+ ]Y(u \otimes e^\alpha) + \partial^{(m_0-1)} H_{i_s}(z)_+ [\text{ad}(T)Y(u \otimes e^\alpha)] \\ &\quad + [\text{ad}(T)Y(u \otimes e^\alpha)] \partial^{(m_0-1)} H_{i_s}(z)_- + Y(u \otimes e^\alpha) [\text{ad}(T) \partial^{(m_0-1)} H_{i_s}(z)_-] \\ &= \partial_z Y(h_{i_s}(-m_0) \vee u \otimes e^\alpha) + [(\text{ad}(T) - \partial_z) \partial^{(m_0-1)} H_{i_s}(z)_+ ]Y(u \otimes e^\alpha) + Y(u \otimes e^\alpha) [(\text{ad}(T) - \partial_z) \partial^{(m_0-1)} H_{i_s}(z)_-]. \end{aligned}$$

Hence we need to prove that

$$[(\text{ad}(T) - \partial_z) \partial^{(m_0-1)} H_{i_s}(z)_+ ]Y(u \otimes e^\alpha) + Y(u \otimes e^\alpha) [(\text{ad}(T) - \partial_z) \partial^{(m_0-1)} H_{i_s}(z)_-] = 0.$$

From the formulas (i) and (iii), it is easy to prove

$$[(\text{ad}(T) - \partial_z) \partial^{(m_0-1)} H_{i_s}(z)_+ ] = 0, [(\text{ad}(T) - \partial_z) \partial^{(m_0-1)} H_{i_s}(z)_-] = 0.$$

Hence (10) holds.

### 3.3. The locality axiom

In this subsection, we check the locality axiom. Namely, we will prove the following formula holds.

$$(z-w)^N [Y(u \otimes e^\alpha, z), Y(v \otimes e^\beta)] = 0, \quad \forall u \otimes e^\alpha, v \otimes e^\beta \in V_Q, z, w \in \mathbb{C}.$$

**Lemma 5:**  $Y(h_i(-1) \otimes 1, z)$  and  $Y(h_j(-1) \otimes 1, w)$  are local, i.e.,

$$(z-w)^N [Y(h_i(-1) \otimes 1, z), Y(h_j(-1) \otimes 1, w)] = 0.$$

**Proof:** Notice that

$$[Y(h_i(-1) \otimes 1, Y(h_j(-1) \otimes 1, w))] = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} [H_i(m) H_j(n)] z^{-m-1} w^{-n-1} = (\alpha_i, \beta_j) \sum_{m \in \mathbb{Z}} m z^{-m-1} w^{-m-1} = (\alpha_i, \beta_j) \partial_w \delta(z-w).$$

By the formula (iv) of Lemma 3, it is easy to prove

$$(z-w)^2 [Y(h_i(-1) \otimes 1, Y(h_j(-1) \otimes 1, w))] = (\alpha_i, \beta_j) (z-w)^2 \partial_w \delta(z-w) = 0.$$

**Lemma 6**<sup>[12]</sup>:  $Y(h_i(-m_i) \otimes 1, z)$  and  $Y(h_j(-n_j) \otimes 1, z)$  are local, i.e.,

$$(z-w)^{m_i+n_j} [Y(h_i(-m_i) \otimes 1, z), Y(h_j(-n_j) \otimes 1, w)] = 0.$$

**Proof:** Since

$$\begin{aligned} Y(h_i(-m_i) \otimes 1, z) \cdot Y(h_j(-n_j) \otimes 1, w) &= Y(h_j(-n_j) \otimes 1, w) \cdot Y(h_i(-m_i) \otimes 1, z) \\ &+ (\alpha_i, \alpha_j) \sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j} \cdot \frac{(m+m_i-1)! (m+m_i+n_j-1)!}{m!(m_i-1)! (m-m_i)!(n_j-1)!} (-m-m_i) a_{n_j}(n_j) \\ &+ (\alpha_i, \alpha_j) \sum_{n=0}^{\infty} z^{-n-m_i-n_j} w^n \cdot \frac{(n+m_i+n_j-1)! (n+n_j-1)!}{(n+n_j)!(m_i-1)! n!(n_j-1)!} (n+n_j) a_{m_i}(m_i), \end{aligned}$$

then

$$\begin{aligned} &Y(h_i(-m_i) \otimes 1, z) \cdot Y(h_j(-n_j) \otimes 1, w) - Y(h_j(-n_j) \otimes 1, w) \cdot Y(h_i(-m_i) \otimes 1, z) \\ &= (\alpha_i, \alpha_j) \left( \sum_{n=0}^{\infty} z^{-n-m_i-n_j} w^n \frac{(n+m_i+n_j-1)!}{(m_i-1)! n!(n_j-1)!} a_{m_i}(m_i) - \sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j} \frac{(m+m_i+n_j-1)!}{(n_j-1)! m!(m_i-1)!} a_{n_j}(n_j) \right) \\ &= (\alpha_i, \alpha_j) \frac{(m_i+n_j-1)!}{(m_i-1)!(n_j-1)!} \left[ \sum_{n=0}^{\infty} z^{-n-m_i-n_j} w^n \cdot \frac{(n+m_i+n_j-1)!}{n!(m_i+n_j-1)!} a_{m_i}(m_i) - \sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j} \frac{(m+m_i+n_j-1)!}{m!(m_i+n_j-1)!} a_{n_j}(n_j) \right] \end{aligned}$$

By the formula (iv) of Lemma 3, we have

$$(z-w)^{m_i+n_j} [Y(h_i(-m_i) \otimes 1, z), Y(h_j(-n_j) \otimes 1, w)] = 0.$$

**Lemma 7:**  $Y(h_i \otimes 1, z)$  and  $Y(1 \otimes e^\alpha, w)$  are local, i.e.,

$$(z-w)^N [Y(h_i \otimes 1, z), Y(1 \otimes e^\alpha, w)] = 0.$$

**Proof:** Since

$$Y(h_i \otimes 1, z) = H_i(z) = \sum_{m \in \mathbb{Z}} H_i(m) z^{-m-1},$$

then

$$[H_i(m), Y(1 \otimes e^\alpha, w)] = w^m (\alpha_i, \alpha) Y(1 \otimes e^\alpha, w),$$

and

$$[Y(h_i \otimes 1, z) Y(1 \otimes e^\alpha, w)] = \delta(z-w) Y(1 \otimes e^\alpha, w).$$

Therefore,

$$(z-w) [Y(h_i \otimes 1, z) Y(1 \otimes e^\alpha, w)] = 0.$$

**Lemma 8:**  $Y(1 \otimes e^\alpha, z)$  and  $Y(1 \otimes e^\beta, w)$  are local, i.e.

$$(z-w)^N [Y(1 \otimes e^\alpha, z), Y(1 \otimes e^\beta, w)] = 0.$$

**Proof:** Since

$$Y(1 \otimes e^\alpha, z) = E^+(\alpha, z) E^-(\alpha, z) x(\alpha, z),$$

We have

$$\begin{aligned} Y(1 \otimes e^a, z)Y(1 \otimes e^b, w) &= x(\alpha, z)x(\beta, w)E^+(\alpha, z)E^-(\alpha, z)E^+(\beta, z)E^-(\beta, z) \\ &= z^{-(\alpha, \beta)}(z-w)^{(\alpha, \beta)}x(\alpha, z)x(\beta, w)E^+(\alpha, z)E^+(\beta, z)E^-(\alpha, z)E^-(\beta, z). \end{aligned}$$

By the same way, we get

$$\begin{aligned} Y(1 \otimes e^b, z)Y(1 \otimes e^a, w) &= x(\beta, w)x(\alpha, z)E^+(\beta, w)E^-(\beta, w)E^+(\alpha, z)E^-(\alpha, z) \\ &= w^{-(\alpha, \beta)}(w-z)^{(\alpha, \beta)}x(\beta, w)x(\alpha, z)E^+(\alpha, z)E^+(\beta, z)E^-(\alpha, z)E^-(\beta, z). \end{aligned}$$

From the definition of mapping  $\varepsilon$  [13], we have

$$z^{-(\alpha, \beta)}(z-w)^{(\alpha, \beta)}x(\alpha, z)x(\beta, w) = w^{-(\alpha, \beta)}(w-z)^{(\alpha, \beta)}x(\beta, w)x(\alpha, z),$$

$$\text{i.e. } (z-w)[Y(1 \otimes e^a, z), Y(1 \otimes e^b, w)] = 0.$$

**Lemma 9**<sup>[13]</sup> : If  $a(z)$ ,  $b(z)$  and  $c(z)$  are pairwise mutually local fields, then  $a(z)_{(n)}b(z)$  and  $c(z)$  are mutually local fields for all  $n \in \mathbb{Z}$  (resp.  $n \in \mathbb{Z}_+$ ). In particular  $a(z)b(z)$  and  $c(z)$  are mutually local fields provided that  $a(z)$ ,  $b(z)$  and  $c(z)$  are.

**Lemma 10:** For any  $u \otimes e^a, v \otimes e^b \in V_Q$ , there exists a nonnegative integer  $N$  satisfies

$$(z-w)^N[Y(u \otimes e^a, z), Y(v \otimes e^b, w)] = 0.$$

**Proof:** From the Lemma 5-8, we know that  $Y(h_i(-1) \otimes 1, z)$ ,  $Y(h_j(-m) \otimes 1)$  and  $Y(1 \otimes e^a)$  are pairwise mutually local fields. Then by the Lemma 9, we can easily prove Lemma 10.

From the fact that we have checked that  $(V, Y(v \otimes e^a, z))$  satisfies the three axioms. It follows

**Theorem 2:**  $(V, Y(v \otimes e^a, z))$  is a vertex algebra.

#### 4. THE CONFORMAL VECTOR OF THE VERTEX ALGEBRA

**Definition 3:** A conformal vector of a vertex algebra  $V$  is an even vector  $v$  such that the corresponding field  $Y(v, z) = \sum_{n \in \mathbb{Z}} L_n z^{(-n-2)}$  is a Virasoro field with central charge  $c$  which has the following properties:

- (a)  $L_{-1} = T$ ,
- (b)  $L_0$  is diagonalizable on  $V$ .

The number  $c$  is called the central charge of  $v$ .

Let  $A = ((\alpha_i, \alpha_j))^{-1} = (a_{ij})$ , then  $A$  is a symmetric matrix. Let

$$\begin{aligned} v &= \frac{1}{2} \sum_{j,k=1}^n a_{jk} H_j(-1) H_k(-1) |0\rangle, \text{ Then} \\ Y(v, z) &= \frac{1}{2} \sum_{j,k=1}^n a_{jk} : H_j(z) H_k(z) : = \sum_{n \in \mathbb{Z}} L_n z^{(-n-2)}. \end{aligned}$$

So we have

$$\begin{aligned} L_0 &= \frac{1}{2} \sum_{j,k} a_{jk} H_j(0) H_k(0) + \frac{1}{2} \sum_{j,k} \sum_{n>0} a_{jk} (H_j(-n) H_k(n) + H_j(n) H_k(-n)); \\ L_{-1} &= \frac{1}{2} \sum_{j,k} \sum_{n \geq 0} a_{jk} (H_j(-n-1) H_k(n) + H_k(-n-1) H_j(n)); \\ L_m &= \frac{1}{2} \sum_{j,k} \sum_{n \in \mathbb{Z}} a_{jk} H_j(n) H_k(m-n), m \neq 0. \end{aligned}$$



In the following, we will proof that the  $L_m$  satisfies the above properties (a) and (b).

By the define of  $L_0$ , we have

$$L_0(1 \otimes e^\beta) = \frac{1}{2} \sum_{j,k} a_{jk} H_j(0) H_k(0) (1 \otimes e^\beta) = \frac{1}{2} (\beta, \beta) (1 \otimes e^\beta) \quad (9)$$

$$\begin{aligned} L_0(h_i(-m) \otimes 1) &= \frac{1}{2} \sum_{j,k} a_{jk} (H_j(-n) H_k(n) + H_j(n) H_k(-n)) h_i(-m) \otimes 1 \\ &= \frac{1}{2} \sum_{j,k} a_{jk} (m(\alpha_k, \alpha_i) h_j(-m) \otimes 1 + m(\alpha_j, \alpha_i) h_k(-m) \otimes 1) \\ &= m h_i(-m) \otimes 1, \end{aligned} \quad (10)$$

Then by the Inductive method, so

$$L_0(h_1(-m_1) \vee \cdots \vee h_s(-m_s) \otimes e^\beta) = (m_1 + \cdots + m_s + \frac{1}{2} (\beta, \beta)) (h_1(-m_1) \vee \cdots \vee h_s(-m_s) \otimes e^\beta).$$

Thus  $L_0$  satisfies the above properties (a) i.e.  $L_0$  is diagonalizable on  $v$ .

By the define of  $L_{-1}$ , we have

$$L_{-1}(1 \otimes e^\alpha) = \alpha(-1)(1 \otimes e^\alpha),$$

and

$$L_{-1}(h(-m) \otimes 1) = m(h(-m) \otimes 1).$$

By the define of  $T$ , we proof that  $L_{-1}$  satisfies the above properties (b) i.e.  $L_{-1} = T$ .

The field  $Y(v, z) = \sum_{n \in \mathbb{Z}} L_n z^{(-n-2)}$  is a Virasoro field and the  $(V, Y(v \otimes e^\alpha, z))$  is a conformal vertex algebra.

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