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Products of $L_2(11)$ by alternating groups

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ABSTRACT

In this note, we will find the structure of the finite simple groups G with two subgroups A and B such that $G = AB$, where A is a simple group and B is isomorphic to the projective special linear group $L_2(11)$.

Keywords: *Simple group, Factorization, Alternating group, Projective special linear group.*

Mathematics Subject Classification (2010): *20D40.*

1. INTRODUCTION

Let G be a group with subgroups A and B. If $G = AB$, then G is called factorizable group and $G = AB$ is called a factorization of G. Sometimes we say that G is a product of two subgroups A and B. It is an interesting problem to know the groups with proper factorization. Of course not every group has a proper factorization, for example an infinite group with all proper subgroups finite has no proper factorization, $L_2(13)$ and also the Janko simple group J_1 of order 175560 have no proper factorization.

A factorization $G = AB$ is called maximal if both factors A and B are maximal subgroups of G. In [12] all the maximal factorizations of all the finite simple groups and their automorphism groups are found. In [17], all the factorizations of the alternating and symmetric groups are found with both factors simple.

Here we quote some results concerning the alternating groups in a factorization. In [15], factorizable groups where one factor is a non-abelian simple group and the other factor is isomorphic to the alternating group on 5 letters are classified. Also in [4], the structure of finite factorizable group with one factor a simple group and the other factor isomorphic to the symmetric group on 6 letters is determined. In [5], the structure of factorizable groups $G = AB$ where $A \cong A_7$ and $B \cong S_n$ was given. In ([6], the structure of the finite simple factorizable groups $G = AB$ such that A is a non-abelian simple group and $B \cong A_7$, the symmetric group on seven letters is classified. In [13], the structure of products of simple groups with alternating group A_8 of degree eitht is determined. As a development of the topics, we determined the structure of products of an alternating group with $L_2(11)$.

2. PRELIMINARY RESULTS

In this section we obtain results which are needed in the proof of our main theorem. Suppose Ω is a set of cardinality m and G is a k-homogeneous, $1 \le k \le m$, group on Ω.The following Lemma is well-known.

Lemma 2.1: Let G be a k-homogeneous permutation group on a set Ω; $0 \le k \le |\Omega|$. Let H be a k-homogeneous subgroup of G. Then $G = G_\delta H$, where δ is a k-subset of Ω, and G_δ is its stabilizer.

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If H is a k-homogeneous subgroup of G, then from [7] we get that the orders of subgroups of $L_2(11)$ are: 1, 2, 3, 4, 5, 6, 10, 11, 12, 55, 60, 660 and the orders of subgroups of $L_2(13)$ are: 1, 2, 3, 4, 7, 12, 13, 14, 26, 39, 1092. Thus the indexes of subgroups of $L_2(11)$ are: 1, 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660. It is well-known that $L_2(11)$ has a 2-transitive action on 12 points [2]. Since we need factorizations of the alternating group involving $L_2(11)$, hence using [12], we will prove the following results.

Lemma 2.2: Let A_m denote the alternating group of degree m. If $A_m = AB$ is a non-trivial factorization of A_m where A a non-abelian simple group of A_m and $B \cong L_2(11)$, then one of the following cases occurs:

(a) $A_m = A_{m-1}L_2(11)$, where $m = 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660$.

(b) $A_{12} = A_{10}L_2(11)$.

(c) $A_{11} = A_9L_2(11)$.

(d) $A_{12} = A_9L_2(11)$.

Proof: It is obvious that m is at least 11. By Theorem D of [12], we have that either $m = 6$, 8 or 10 or one of A or B is k-homogeneous on m letters. Since $m = 6$, 8 or 10, A_m does not involve $L_2(11)$ and so we consider the following cases.

Case (i): $A_{m-k} \trianglelefteq A \le S_{m-k} \times S_k$ for some k with $1 \le k \le 5$, and B is k-homogeneous on m letters.

Since A is assumed to be simple we obtain $A_{m-k} = 1$ or A. If $A_{m-k} = 1$, then $m-k = 1$ or 2, hence $k = m-1$ or $m-2$. But then from $1 \le k \le 5$ we obtain $2 \le m \le 6$ or $3 \le m \le 7$, a contradiction because $m \ge 11$. Therefore $A = A_{m-k}$ and $B \cong A_8$ is k-homogeneous on m letters, $1 \le k \le 5$. If $k = 1$, then by Lemma 2.1, the size of the set Ω on which $L_2(11)$ can act transitively is as stated in the Lemma and all the factorizations in case (a) occur. If $k \ge 2$, then $m = 12$, and so $A_{12} = A_{10}L_2(11)$. This is the Case (b).

Case (ii): $A_{m-k} \trianglelefteq B \le S_{m-k} \times S_k$ for some k with $1 \le k \le 5$, and A is k-homogeneous on m letters.

Since $B \cong L_2(11)$ we obtain $A_{m-k} = 1$ or B and so $m-k =$ 1, 2 or 11. From $1 \le k \le 5$, we have $2 \le m \le 6$, $3 \le m \le 9$ or $12 \le m \le 16$. Therefore, we know that only $m = 12, 13, 14, 15$ or 16 are possible which is correspond to $k = 1, 2, 3, 4, 5$ respectively. We have from Theorem 4.11 and page 197 of [2], and [11], that the possible solutions for (m, k) are (11, 2), (12, 3). Thus $A_{11} = A_9L_2(11)$ and $A_{12} = A_9L_2(11)$.

3. MAIN RESULT

To find the structure of the factorizable simple groups $G = AB$ with A simple and $B \cong L_2(11)$, we need to know about the primitive groups of certain degrees which are equal to the indices of subgroups in $L_2(11)$. From [8], we list the primitive permutation groups of degree n less than 1000 as Table 1.

Table - 1: Non-abelian simple primitive groups of degree less than 660.

degree	group
11	A_{11}, M_{11}
12	$A_{12}, L_2(11), M_{11}, M_{12}$
55	$A_{55}, A_{11}, L_2(11)$
60	A_{60}
66	$A_{66}, A_{12}, M_{11}, M_{12}$

110	A_{110}
132	$A_{132}, L_3(8)$
165	A_{165}, A_{11}, M_{11}
220	A_{220}, M_{12}
330	A_{330}
660	A_{660}

Theorem 3.1 Let $G = AB$ is a non-trivial factorization of a simple group G with A a non-abelian simple group and $B \cong L_2(11)$, then one of the following cases occurs:

(a) $A_m = A_{m-1}L_2(11)$, where $m = 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660$.

(b) $A_{12} = A_{10}L_2(11)$.

(c) $A_{11} = A_9 L_2(11)$.

(d) $A_{12} = A_9 L_2(11)$.

(e) $M_{12} = M_{11}L_2(11)$.

Proof: Assume that $G = AB$ is a non-trivial factorization of a simple group G with A a non-abelian simple group and $B \cong L_2(7)$. If M is a maximal subgroup of G containing A, then $G = MB$, hence $\langle |G:M| \big| |B:M\cap B| \rangle$. Since $d = |B : B \cap M|$ is equal to the index of a subgroup of A_8, therefore G is primitive permutation group of degree d. We know that $d = 1, 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660$. It is easy to see that $d \neq 1$. If G is an alternating group, then from Lemmas 2.1 and 2.2, we have that the cases (a) and (b) is as in the Theorem. Using Table 1, we only consider the following groups: M_{11}, M_{12} and $L_3(8)$.

Let M be a maximal subgroup of G containing A.

If $G = M_{11}$, then $d = |G:M| = 11, 12, 66, 165.$ According to (Conway et al, 1985), we have the foollowing. If $d = 11$ we get $M \cong A_6 \cdot 2$ and so $A = A_6$. Therefore $M_{11} = A_6 L_2(11)$. Order consideration, the subgroup of order 30 belongs to both A_6 and $L_2(11)$, a contradiction since A_6 has no subgroup of order 30. If $d = 12$, then $M \cong L_2(11)$ and so $A = L_2(11)$, which means that $M_{11} = L_2(11)L_2(11) = L_2(11)$, a contradiction. If $d = 66$, then $M \cong S_5$ and so $A \cong A_5$. Hence $M_{11} = A_5 L_2(11)$. On the other hand, A_5 is a subgroup of $L_2(11)$ and so $M_{11} = L_2(11)$, a contradiction. If $d = 165$, then $M \cong 2:S_4$ and so $A \cong S_4$, but S_4 is soluble, a contradiction.

If $G = M_{12}$, then $d = 1266220$. According to [1], we have the following. If $d = 12$, then $M \cong M_{11}$ and so $A \cong M_{11}$. Hence $M_{12} = M_{11}L_2(11)$. Since the subgroup of order 55 is both contained in $L_2(11)$ and M_{11}, then this is the case. If $d = 66$, then $M \cong A_6 \cdot 2^2$ and so $A = A_6$. Order consideration rules out the case. If $d = 220$, then $M \cong 3^2 : 2S_4$. We rule out this case.

If $G = L_3(8)$, then there is no subgroup of index 132 and so we rule out this case.

This completes the proof of the Theorem.

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