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DUO TERNARY SEMIGROUPS

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ABSTRACT

In this paper the terms left duo teranry semigroup, right duo teranry semigroup, duo teranry semigroup are introduced. it is proved that a ternary semigroup T is a duo teranry semigroup if and only if $xT^1T^1 = T^1T^1x = T^1xT^1$ for all $x \in T$. Further it is proved that every commutative / quasi commutative ternary semigroup is a duo ternary semigroup. If A is an ideal of a ternary semigroup T and $a \in T$, then 1) $A_l(a) = \{ x \in T : xua \in A \}$ is a left ideal of T for all $u \in T$. 2) $A_r(a) = \{ x \in T : aux \in A \}$ is a right ideal of T for all $u \in T$. If A is an ideal of a duo ternary semigroup T and $a \in T$, then 1) $A_l(a) = \{ x \in T : xua \in A \}$ is an ideal of T for all $u \in T$. 2) $A_r(a) = \{x \in T: aux \in A\}$ is an ideal of T for all $u \in T$. It is proved that if A is an ideal of a duo ternary semigroup T, then 1) $abc \in A$ if and only if $< a > < b > < c > \subseteq A$ for all a, b, $c \in T$. 2) $a_1a_2 .....a_n \in A$ if and only if $< a_1 > < a_2 > < a_3 >........< a_n > \subseteq A$ for all $a_1 , a_2 , a_3 ,. . . .a_n \in T$. 3) $a^n \in A$ if and only if $< a >^n \subseteq A$ for all $a \in A$. 4) $< abc > = < a > < b > < c >$ for all a, b, $c \in T$. 5) $< a^n > = < a >^n$ for all $a \in T$. Further it is proved, if A is an ideal of duo ternary semigroup T then 1) $A_4 = \{x : < x >^n \subseteq A$ for some odd natural number n $\}$ is the minimal semiprime ideal of T containing A. 2) $A_2 = \{x \ \ T : x^n \in A$ for some odd natural number n$\}$ is the minimal completely semiprime ideal of T containing A. It is proved that, 1) An ideal A of a duo ternary semigroup T is completely semiprime if and only if A is semiprime. 2) Every prime ideal P minimal relative to containing an ideal A of a duo ternary semigroup T is completely prime. It is also proved that, if T is a duo ternary semigroup and A is an ideal of T, then $A_1=A_2=A_3=A_4$. It is proved that 1) in a duo semi group T, the following are equivalent 1) T is a strongly archimedean ternary semigroup. 2) T is an archimedean ternary semigroup. 3) T has no proper completely prime ideals. 4) T has no proper completely semiprime ideals. 5) T has no proper prime ideals. 6) T has no proper semiprime ideals. Further it is proved that, if T is a duo ternary semigroup, then 1) $S = \{a \in T: \sqrt{< a >} \neq T\}$ is either empty or prime ideal. 2) T\S is either empty or an archemedian ternary sub semigroup of T. It is proved that, if T is a duo ternary semigroup and contains a nontrivial maximal ideal then T contains semisimple elements. Finally, it is proved that, in a duo archimedian ternary semigroup T, an ideal M is maximal if and only if M is trivial. Also if $T = T^3$, then T has no maximal ideals.

1. INTRODUCTION

Ternary semigroups was introduced by Santiago. M. L. And Bala S.S., [9] as a generalization of semigroup. KRULL proved that the nil-radical of an ideal A in a commutative ring is equal to the intersection of all minimal prime ideals containing A. SATYANARAYANA [7], [8] has developed some literature on prime ideals and prime radicals for commutative semigroups and obtained KRULL's theorem for commutative semigroups. GIRI and WAZALWAR [4] studied about prime radicals in general semigroups. ANJANEYULU.A [1], [2], [3] made a study on prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals in duo semigroups. SARALA.Y, ANJANEYULU.A, MADHUSUDHANA RAO [5], [6] studied about the prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals, prime radicals in general ternary semigroups. In this paper we study about the prime ideals, completely prime ideals, semiprime ideals and completely semiprime ideals, prime radicals and generalize the results obtained by ANJANEYULU. A in duo ternary semigroups.

2. PRILIMINARIES

Definition 2.1: Let T be a non-empty set. Then T is said to be a ***ternary semigroup*** if there exist a mapping from T×T×T to T which maps $(x_1, x_2, x_3) \to [x_1x_2x_3]$ satisfying the condition:

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$\left[\left(x_1x_2x_3\right)x_4x_5\right]=\left[x_1\left(x_2x_3x_4\right)x_5\right]=\left[x_1x_2\left(x_3x_4x_5\right)\right]\ \forall\ x_i\ \in$ T, $1\le i\le 5$.

Note 2.2: For the convenience we write $x_1x_2x_3$ instead of $\left[x_1x_2x_3\right]$

Note 2.3: Let T be a ternary semigroup. If A,B and C are three subsets of T, we shall denote the set ABC = $\left\{abc : a\in A, b\in B, c\in C\right\}$.

Definition 2.4: A ternary semigroup T is said to be ***commutative*** provided *abc* = *bca* = *cab* = *bac* = *cba* = *acb* for all *a, b, c* $\in$ T.

Definition 2.5: A ternary semigroup T is said to be ***quasi commutative*** provided for each *a, b, c* $\in$ T, there exists a natural number *n* such that *abc* = $b^n ac$ = *bca* = $c^n ba$ = *cab* = $a^n cb$.

Definition 2.6: A nonempty subset A of a ternary semigroup T is said to be ***left ternary ideal*** or ***left ideal*** of T if *b, c* $\in$ T, *a* $\in$ A implies *bca* $\in$ A.

Note 2.7: A nonempty subset A of a ternary semigroup T is a left ideal of T if and only if TTA $\subseteq$ A.

Definition 2.8: A nonempty subset of a ternary semigroup T is said to be a ***lateral ternary ideal*** or simply ***lateral ideal*** of T if *b, c* $\in$ T , *a* $\in$ A implies *bac* $\in$ A.

Note 2.9: A nonempty subset of A of a ternary semigroup T is a lateral ideal of T if and only if TAT $\subseteq$ A.

Definition 2.10: A nonempty subset A of a ternary semigroup T is a ***right ternary ideal*** or simply ***right ideal*** of T if *b, c* $\in$ T , *a* $\in$ A implies *abc* $\in$ A

Note 2.11: A nonempty subset A of a ternary semigroup T is a right ideal of T if and only if ATT $\subseteq$ A.

Definition 2.12: A nonempty subset A of a ternary semigroup T is a ***two sided ternary ideal*** or simply ***two sided ideal*** of T if *b, c* $\in$ T , *a* $\in$ A implies *bca* $\in$ A, *abc* $\in$ A.

Note 2.13: A nonempty subset A of a ternary semigroup T is a two sided ideal of T if and only if it is both a left ideal and a right ideal of T .

Definition 2.14: A nonempty subset A of a ternary semigroup T is said to be ***ternary ideal*** or simply an ***ideal*** of T if *b, c* $\in$ T , *a* $\in$ A implies *bca* $\in$ A, *bac* $\in$ A, *abc* $\in$ A.

Note 2.15: A nonempty subset A of a ternary semigroup T is an ideal of T if and only if it is left ideal, lateral ideal and right ideal of T .

Definition 2.16: An ideal A of a ternary semigroup T is said to be a ***proper ideal*** of T if A is different from T.

Definition 2.17: An ideal A of a ternary semigroup T is said to be a ***trivial ideal*** provided T\ A is singleton.

Definition 2.18: An ideal A of a ternary semigroup T is said to be a ***maximal ideal*** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

Theorem 2.19: If T is a ternary semigroup with unity 1 then the union of all proper ideals of T is the unique maximal ideal of T.

Definition 2.20: An ideal A of a ternary semigroup T is said to be ***a principal ideal*** provided A is an ideal generated by $\left\{a\right\}$ for some *a* $\in$ T. It is denoted by J (*a*) (or) < *a* >.

Notation 2.21: Let T be a ternary semigroup. If T has an identity, let T^1 = T and if T does not have an identity , let T^1 be the ternary semigroup T with an identity adjoined usually denoted by the symbol 1.

Notation 2.22: Let T be a ternary semigroup. if T has a zero, let T^0 = T and if T does not have a zero , let T^0 be the ternary semigroup T with zero adjoined usually denoted by the symbol 0.

Definition 2.23: An ideal A of a ternary semigroup T is said to be a ***completely prime ideal*** of T provided $x, y, z \in$ T and $xyz \in$ A implies either $x \in$ A or $y \in$ A or $z \in$ A.

Definition 2.24: An ideal A of a ternary semigroup T is said to be a ***prime ideal*** of T provided X,Y,Z are ideals of T and XYZ $\subseteq$ A $\Rightarrow$ X $\subseteq$ A or Y $\subseteq$ A or Z $\subseteq$ A.

Definition 2.25: If A is an ideal of a ternary semigroup T, then the intersection of all prime ideals of T containing A is called ***prime radical*** or simply ***radical*** of A and it is denoted by $\sqrt{A}$ or *rad* A.

Definition 2.26: If A is an ideal of a ternary semigroup T, then the intersection of all completely prime ideals of T containing A is called ***completely prime radical*** or simply ***complete radical*** of A and it is denoted by *c.rad* A.

Corollary 2.27: If $a \in \sqrt{A}$, then there exist a positive integer n such that $a^n \in$ A for some odd natural number $n \in$ N.

Corollary 2.28: If A is an ideal of a commutative ternary semigroup T, then *rad* A = *c.rad* A.

Definition 2.29: An element a of ternary semigroup T is said to be ***left identity*** of T provided $aat = t$ for all $t \in$ T.

Note 2.30: Left identity element a of a ternary semigroup T is also called as ***left unital element.***

Definition 2.31: An element a of a ternary semigroup T is said to be a *lateral identity* of T provided $ata = t$ for all $t \in$ T.

Note 2.32: Lateral identity element a of a ternary semigroup T is also called as ***lateral unital element.***

Definition 2.33: An element a of a ternary semigroup T is said to be a ***right identity*** of T provided $taa = t \ \forall \ t \in$ T.

Note 2.34: Right identity element a of a ternary semigroup T is also called as ***right unital element.***

Definition 2.35: An element a of a ternary semigroup T is said to be a ***two sided identity*** of T provided $aat = taa = t \ \forall \ t \in$ T.

Note 2.36: Two-sided identity element of a ternary semigroup T is also called as ***bi-unital element***.

Definition 2.37: An element a of a ternary semigroup T is said to be an ***identity*** provided $aat = taa = ata = t \ \forall \ t \in$ T.

Note 2.38: An identity element of a ternary semigroup T is also called as ***unital element.***

Note 2.39: An element a of a ternary semigroup T is an ***identity*** of T iff a is left identity, lateral identity and right identity of T.

Definition 2.40: An ideal A of a ternary semigroup T is said to be a ***proper ideal*** of T if A is different from T.

Definition 2.41: An ideal A of a ternary semigroup T is said to be a ***trivial ideal*** provided T\ A is singleton.

Definition 2.42: An ideal A of a ternary semigroup T is said to be a ***maximal ideal*** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

Definition 2.43: An element a of a ternary semigroup T is said to be ***semisimple*** if $a \in \langle a \rangle^3$ i.e. $\langle a \rangle^3 = \langle a \rangle$.

Definition 2.44: A ternary semigroup T is called ***semisimple*** ternary ***semigroup*** provided every element in T is semisimple.

Theorem 2.44: If T is a ternary semigroup with unity 1 then the union of all proper ideals of T is the unique maximal ideal of T.

Definition 2.45: An ideal A of a ternary semigroup T is said to be a ***completely semiprime ideal*** provided $x \in$ T, $x^n \in$ A for some odd natural number n >1 implies $x \in$ A.

Definition 2.46: An ideal A of a ternary semigroup T is said to be ***semiprime ideal*** provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number n implies X ⊆ A.

Theorem 2.47: An ideal Q of ternary semigroup T is a semiprime ideal of T if and only if $\sqrt{Q}$ = Q.

Notation 2.48: If A is an ideal of a ternary semigroup T, then we associate the following four types of sets.
A_1 = The intersection of all completely prime ideals of T containing A.
A_2 = $\{x \in T: x^n \in A$ for some odd natural numbers $n\}$
A_3 = The intersection of all prime ideals of T containing A.
A_4 = $\{x \in T: < x >^n \subseteq A$ for some odd natural number $n\}$

Theorem 2.49: If A is an ideal of a ternary semigroup T, then A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

3. DUO TERNARY SEMIGROUPS

Definition 3.1: A ternary semigroup T is said to be a ***left duo ternary semigroup*** if every left ideal of T is both right and lateral ideal of T.

Definition 3.2: A ternary semigroup T is said to be a ***right duo ternary semigroup*** if every right ideal of T is both left and lateral ideal of T.

Definition 3.3: A ternary semigroup T is said to be a ***duo ternary semigroup*** if it is both a left duo ternary semigroup and a right duo ternary semigroup.

Theorem 3.4: If T is a ternary semigroup and $x \in$ T, then T Tx = { pqx : $p, q \in$ T } is a left ideal of T.

Proof: Let $a \in$ TTx and $s, t \in$ T. Now $a \in$ TTx, implies $a = pqx$ for some $p, q \in$ T.

Now $p, s, t \in$ T and T is a ternary semigroup, implies that $stp \in$ T $\Rightarrow (stp)qx \in$ TTx

$\Rightarrow st(pqx) \in$ TTx. Thus $sta \in$ TTx. Therefore TTx is a left ideal of T.

Theorem 3.5: If T is a ternary semigroup and $x \in$ T, then TxT = { pxq : $p, q \in$ T } is a left and right ideal of T.

Proof: Let $a \in$ TxT and $s, t \in$ T. Now $a \in$ TxT, implies $a = pxq$ for some $p, q \in$ T.

Now $s, t, q, p \in$ T and T is a ternary semigroup, implies that $stp, qst \in$ T $\Rightarrow stpxq \in$ TxT and $pxqst \in$ TxT $\Rightarrow sta$, $ast \in$ TTx. Therefore TxT is a left and right ideal of T.

Theorem 3.6: If T is a ternary semigroup and $x \in$ T, then xTT = { xpq : $p, q \in$ T } is a right ideal of T.

Proof: Let $a \in x$TT and $s, t \in$ T. Now $a \in x$TT, implies $a = xpq$ for some $p, q \in$ T.

Now $q, s, t \in$ T and T is a ternary semigroup, implies that $qst \in$ T $\Rightarrow xp(qst) \in x$TT

$\Rightarrow (xpq)st \in x$TT. Thus $ast \in x$TT. Therefore xTT is a right ideal of T.

Theorem 3.7: A ternary semigroup T is a duo ternary semigroup if and only if $xT^1T^1 = T^1T^1x = T^1xT^1$ for all $x \in$ T.

Proof: Suppose that T is a duo ternary semigroup and $x \in$ T.

Let $t \in xT^1T^1$. Then $t = xyz$ for some $y, z \in T^1$. By theorem 3.4 T^1T^1x is a left ternary ideal of T and T is left duo ternary semigroup, implies that T^1T^1x is a ternary ideal of T. So $x \in T^1T^1x$ is a ternary ideal $\Rightarrow xyz \in T^1T^1x \Rightarrow t \in T^1T^1x$.

Therefore $xT^1T^1 \subseteq T^1T^1x$.

Similarly, we get $xyz \in T^1xT^1 \Rightarrow t \in T^1xT^1$. Therefore $xT^1T^1 \subseteq T^1x T^1$.

Similarly we can prove that $x T^1T^1 \subseteq T^1T^1x$ and $xT^1T^1 \subseteq T^1xT^1$ Therefore $xT^1T^1 = T^1T^1x$ for all $x \in$ T.

Hence $xT^1T^1 = T^1T^1x = T^1xT^1$ for all $x \in$ T.

Conversely suppose that $xT^1T^1 = T^1T^1x = T^1xT^1$ for all $x \in T$.

Let A be a left ternary ideal of T. Let $x \in A$, $y, z \in T^1$. Then $xyz \in xT^1T^1 = T^1T^1x \Rightarrow xyz = tux$ for some $t, u \in T^1$.

Now $x \in A$, $t, u \in T^1$, A is a left ternary ideal of T $\Rightarrow tux \in A \Rightarrow xyz \in A$. Therefore A is a right ternary ideal of T.

Similarly $yxz \in T^1 xT^1 = T^1T^1x \Rightarrow yxz = pqx$ for some $p, q \in T^1$.

Now $x \in A$, $p, q \in T^1$, A is a left ternary ideal of T $\Rightarrow pqx \in A \Rightarrow yxz \in A$. Therefore A is a lateral ternary ideal of T. and hence A is a ternary ideal of T. Therefore T is a left duo ternary semigroup. Similarly we can prove that T is a right duo ternary semigroup. Hence T is a duo ternary semigroup.

Theorem 3.8: Every commutative ternary semigroup is a duo ternary semigroup.

Proof: suppose that T is a commutative ternary semigroup. Therefore $xT^1T^1 = T^1T^1x = T^1xT^1$ for all $x \in T$. By theorem 3.7, T is a duo ternary semigroup.

Definition 3.9: A ternary semigroup T is said to be ***normal*** if $abT = Tab = aTb$ for all $a, b \in T$.

Theorem 3.10: Every normal ternary semigroup is a duo ternary semigroup.

Proof: Suppose that T is a normal ternary semigroup. Then $abT = Tab = aTb$ for all $a, b \in T$.

Therefore $aT^1T^1 = T^1T^1a = T^1aT^1$ for all $a \in T$. By theorem 3.7, T is a duo ternary semigroup.

Theorem 3.11: Every quasi commutative ternary semigroup is a duo ternary semigroup.

Proof: Suppose that T is a quasi commutative ternary semigroup. Then *for* $a, b, c \in T$, there exists $n \in N$, such that $abc = b^n ac = bca = c^n ba = cab = a^n cb$.

Let A be a left ideal of T. Then $TTA \subseteq A$. Let $a \in A$ and $s, t \in T$.

Since T is a quasi commutative ternary semigroup, there exist a natural number n such that $ast = sta = tas \in TTA \subseteq A$. Therefore $ast, tas \in A$ for all $a \in A$ and $s, t \in S$ and hence A is a right and lateral ideal of T. Therefore T is a left duo ternary semigroup. Simillarly we can prove T is a right duo ternary semigroup. Therefore, Every quasi commutative ternary semigroup is a duo ternary semigroup.

Theorem 3.12: If A is an ideal of a ternary semigroup T and $a \in T$, then $A_l(a) = \{x \in T: xua \in A\}$ is a left ideal of T for all $u \in T$.

Proof: Let $x \in A_l(a)$ and $s, t \in T$ T. Now $x \in A_l(a) \Rightarrow xua \in A$ for all $u \in T$. Since A is an ideal of S, $xua \in A$ and $s, t \in T$ $st(xua) \in A$ and hence $stx \in A_l(a)$. Hence $A_l(a)$ is a left ideal of T.

Theorem 3.13: If A is an ideal of a left duo ternary semigroup T and $a \in T$, then $A_l(a) = \{ x \in T : xua \in A \}$ is an ideal of T for all $u \in T$.

Proof: By theorem 3.12, $A_l(a)$ is a left ideal of T. Since T is left duo ternary semigroup, we have A is right and lateral ideal of T. Hence A is ternary ideal of T.

Theorem 3.14: An ideal A of a ternary semigroup T and $a \in T$, then $A_r(a) = \{ x \in T : aux \in A \}$ is a right ideal of T for all $u \in T$.

Proof: Let $x \in A_r(a)$ and $s, t \in T$. Now $x \in A_l(a) \Rightarrow aux \in A$. Since A is an ideal of T, $aux \in A$ and $s, t \in T \Rightarrow auxst \in A$ and hence $xst \in A_r(a)$. Therefore Ar(a) is a right ideal of T.

Theorem 3.15: If A is an ideal A of a right duo ternary semigroup T and $a \in T$, then $A_r(a) = \{ x \in T : aux \in A \}$ is an ideal of T for all $u \in T$.

Proof: By theorem 3.14, $A_r(a)$ is a right ideal of T. Since T is right duo ternary semigroup, we have A is left and lateral ideal of T. Hence A is ternary ideal of T.

Theorem 3.16: Let A be an ideal of a duo ternary semigroup T and $a, b, c \in$ T. Then $abc \in$ A if and only if $< a >< b >< c > \subseteq$ A.

Proof: Suppose that $< a >< b >< c > \subseteq$ A . Now $abc \in \boldsymbol{< a >< b >< c >} \subseteq$ A and hence $abc \in$ A.

Conversely suppose that $abc \in$ A. Let $t \in < a >< b >< c >$. Then $t = paqbrcs$ for some $p, q, r, s \in$ T. Since T is duo ternary semigroup, By theorem 3.7, $xT^1T^1 = T^1T^1x = T^1xT^1$ for all $x \in$ T. Therefore $paq \in T^1aT^1 = T^1T^1a$, implies that $paq = uva$ for some $u, v \in$ T.

Similarly, $rcs \in T^1cT^1 = cT^1T^1$, implies that $rcs = cyz$ for some $y, z \in$ T. Now $t = paqbrcs = (paq)b(rcs) = (uva)b(cyz) = uv(abc)yz \in < abc > \subseteq$ A. Hence $< a > < b > < c > \subseteq$ A.

Corollary 3.17: Let A be an ideal of a duo ternary semigroup T and $a_1, a_2, a_3, \ldots a_n \in$ T. Then $a_1a_2a_3 \ldots a_n \in$ A if and only if $< a_1 >< a_2 >< a_3 > \ldots < a_n > \subseteq$ A.

Proof: By theorem 3.16, proof is clear.

Corollary 3.18: Let A be an ideal of a duo ternary semigroup T and $a \in$ T. Then for any odd natural number n, $a^n \in$ A if and only if $< a >^n \subseteq$ A.

Proof: The proof follows from corollary 3.17, by taking $a = a_1 = a_2 = a_3 = \ldots = a_n$.

Corollary 3.19: Let T be a duo ternary semigroup and A be an ideal of T.

If $a^n \in$ A for some odd natural number n, then $< ast >^n, < sta >^n, < sat >^n \subseteq$ A for all $s, t \in$ T.

Corollary 3.20: Let A be an ideal of a duo ternary semigroup T . If $a^n \in$ A, for some odd natural number n, then $(ast)^n, (sta)^n, (sat)^n \in$ A for all $s, t \in$ T.

Theorem 3.21: Let T be a duo ternary semigroup and $a, b, c \in$ T. Then $< abc > = < a >< b >< c >$.

Proof: Clearly, $abc \in \boldsymbol{< a >< b >< c >}$ and hence $\boldsymbol{< abc > \subseteq < a >< b >< c >}$.

Let $t \in < a >< b >< c >$. Then $t = paqbrcs$ for some $p, q, r, s \in$ T. Since T is duo ternary semigroup, By theorem 3.7, $xT^1T^1 = T^1T^1x = T^1xT^1$ for all $x \in$ T.

Therefore $paq \in T^1aT^1 = T^1T^1a$, implies that $paq = uva$ for some $u, v \in$ T.

Similarly, $rcs \in T^1cT^1 = cT^1T^1$, implies that $rcs = cyz$ for some $y, z \in$ T.

Now $t = paqbrcs = (paq)b(rcs) = (uva)b(cyz) = uv(abc)yz \in < abc >$. Hence $t \in < abc >$.

Thus $< a > < b > < c > \subseteq < abc >$. Therefore $< abc > = < a >< b >< c >$.

Corollary 3.22: Let T be a duo ternary semigroup and $a_1, a_2, a_3, \ldots a_n \in$ T. Then for any odd natural number n, $< a_1a_2a_3 \ldots a_n > = < a_1 >< a_2 >< a_3 > \ldots < a_n >$.

Proof: By theorem 3.21, proof is clear.

Corollary 3.23: Let T be a duo ternary semigroup and $a \in$ T. Then for any odd natural number n, $< a^n > = < a >^n$.

Proof: The proof follows from corollary 3.22, by taking $a = a_1 = a_2 = a_3 = \ldots = a_n$.

Theorem 3.24: If T is a duo ternary semigroup and $a \in$ T, then the following are equivalent.
1) a is regular .
2) a is left regular .
3) a is right regular.
4) a is semisimple.

Proof: Let T is a duo ternary semigroup and $a \in$ T. Then aTT = TaT = TTa.

(1) ⇒ (2): Suppose that a is regular. Therefore $a = axaya$ for some $x \in$ T. Now $xay \in$TaT = aTT ⇒ $xay = apq$ for some $p, q \in$ T. Therefore a = $aapqa$. Again, $pqa \in$ TTa = aTT ⇒ $pqa = ast$ for some $s, t \in$ T. Hence $a = aapqa = aa\, ast = a^3st$ for some $s, t \in$ T. Thus a is left regular.

(2) ⇒ (3): Suppose that a is left regular. Then $a = a^3st$ for some $s, t \in$ T.

Now T is duo ternary semigroup ⇒ a^3TT = TTa^3. Therefore $a = a^3st \in a^3$TT = TTa^3. Therefore $a = a^3yz$ for some $y, z \in T$ and hence S is right regular.

(3)⇒ (4): Suppose that a is right regular. Then $a = a^3yz$ for some $y, z \in$ T.

Hence $a = a^3yz \in a^3$TT ⊆ $< a^3 >$. Since T is duo, by corollary 3.23, $a = a^3yz \in < a^3 > = < a >^3$. Hence a is semisimple.

(4) ⇒ (1): Suppose that a is semisimple. Then $a \in < a >^3 \Rightarrow a = paqaras$ for some $p, q, r, s \in$ T. Since T is duo, TaT = aTT = TTa. Hence $paq \in a$TT and $ras \in$ TTa. Therefore $paq = alm$ and $ras = jka$ for some $j, k, l, m \in$ T. Therefore $a = paqaras = almajka = axa$ where $x = lmajk$ and hence $axaxa = axa = a$. Thus a is regular in T.

Theorem 3.25: If A is an ideal of duo ternary semigroup T then
$A_4 = \{x : < x >^n \subseteq A$ for some odd natural number $n\}$ is the minimal semiprime ideal of T containing A.

Proof: Clearly A ⊆ A_4 and hence A_4 is nonempty subset of T. Let $x \in A_4$ and $s, t \in$ T.
Since $x \in \mathbf{A_4}$, $< x >^n \subseteq$ A for some odd natural number n. Now $< xst >^n \subseteq < x >^n \subseteq$ A, $< stx >^n \subseteq < x >^n \subseteq$ A and $< sxt >^n \subseteq < x >^n \subseteq$ A. Therefore, $xst, stx, sxt \in A_4$. Therefore $\mathbf{A_4}$ is an ideal of T containing A. Let $x \in$ T such that $< x >^3 \subseteq \mathbf{A_4}$. Then $<< x >^3>^n \subseteq$ A ⇒ $< x >^{3n} \subseteq$ A ⇒ $x \in \mathbf{A_4}$. Therefore $\mathbf{A_4}$ is semiprime ideal of T containing A. Let Q be a semiprime ideal of T containing A. Let $x \in \mathbf{A_4}$. Then $< x >^n \subseteq$ A ⊆ Q for some odd natural number n. Since Q is a semiprime ideal of S, $< x >^n \subseteq$ Q for some odd natural number n.

⇒ $x \in$ Q. Therefore $\mathbf{A_4} \subseteq$ Q and hence $\mathbf{A_4}$ is the minimal semiprime ideal of T containing A.

Theorem 3.26: If A is an ideal of duo ternary semigroup T then $A_2 = \{x \in T: x^n \in A$ for some odd natural number $n\}$ is the minimal completely semiprime ideal of T containing A.

Proof: Clearly A ⊆ $\mathbf{A_2}$ and hence $\mathbf{A_2}$ is nonempty subset of T. Let $x \in \mathbf{A_2}$ and $s, t \in$ T.

Since $x \in \mathbf{A_2}$, $x^n \in$ A for some odd natural number n. Now $(xst)^n \in$ A, $(stx)^n \in$ A and $(sxt)^n \in$ A.

Hence $xst, stx, sxt \in \mathbf{A_2}$. Therefore $\mathbf{A_2}$ is an ideal of T containing A. Let $x \in$ T such that $x^3 \in \mathbf{A_2}$. Then $x^{3n} \in \mathbf{A}$ for some odd natural number n and hence $x \in \mathbf{A_2}$. Therefore $\mathbf{A_2}$ is a completely semiprime ideal of T containing A. Let P be a completely semiprime ideal of T containing A. Let $x \in \mathbf{A_2}$. Then $x^n \in$ A ⊆ P for some odd natural number n. Since P is a completely semiprime ideal of S, $x^n \in$ P ⇒ $x \in$ P. Therefore $\mathbf{A_2} \subseteq$ P and hence $\mathbf{A_2}$ is the minimal completely semiprime ideal of T containing A.

Theorem 3.27: If A is an ideal of a duo ternary semigroup T. Then $A_2 = A_4$.

Proof: By theorem 2.49, we have $A_4 \subseteq A_2$. Let $x \in A_2$. Then $x^n \in$ A for some odd natural number n. By Corrollary 3.18, we get $< x >^n \subseteq$ A and hence $x \in A_4$. Therefore $A_2 \subseteq A_4$ and hence $A_2 = A_4$.

Theorem 3.28: If A is semiprime ideal of a duo ternary semigroup T then A is completely semiprime.

Proof: Let $x \in$ T and $x^3 \in$ A. $x^3 \in$ A. By Corrollary 3.18, $< x >^3 \subseteq$ A. Since A is semiprime, $< x >^3 \subseteq$ A and hence $x \in$ A. Therefore A is completely semiprime.

Theorem 3.29: An ideal A of a duo ternary semigroup T is completely semiprime if and only if A is semiprime.

Proof: Suppose that A is completely semiprime ideal of T. Clearly, A is semiprime ideal of T. Conversely suppose that A is semi prime ideal. By Corollary 3.28, A is completely semiprime ideal of S.

Theorem 3.30: Every prime ideal P minimal relative to containing an ideal A of a duo ternary semigroup T is completely prime.

Proof: Let S be a subternary semigroup of T generated by T\P. First we show that A $\cap$ S =$\emptyset$. If A $\cap$ S $\neq \emptyset$, then there exists $x_1, x_2, \ldots, x_n \in$ S\P such that $x_1x_2x_3\ldots x_n \in$ A where n is an odd natural number *n*. By corollary 3.17, $< x_1 > < x_2 > \ldots\ldots < x_n > \in A \subseteq P$. Since P is a prime ideal, we have $< x_i > \subseteq P$ for some *i*. It is a contradiction.

Thus A $\cap$ S =$\emptyset$. Consider Σ = {B: B is an ideal of T containing A such that B $\cap$ S =$\emptyset$}. Since A $\in \Sigma$, Σ is nonempty.

Now Σ is a poset under set inclusion and satisfies the hypothesis of Zorns Lemma. Thus by Zorn's lemma, Σ contains a maximal element say M. Let X and Y be two ideals in T such that XYZ $\subseteq$ M.

If X $\nsubseteq$ M, Y $\nsubseteq$ M and Z $\nsubseteq$ M, then M $\cup$ X, M $\cup$ Y and M $\cup$ Z are ideals of T containing M properly and hence by the maximality of M, we have (M $\cup$ X) $\cap$T $\neq \emptyset$, (M $\cup$ Y) $\cap$T $\neq \emptyset$ and (M $\cup$ Z) $\cap$T $\neq \emptyset$. Since M $\cap$ T = $\emptyset$, we have X $\cap$ T $\neq \emptyset$, Y $\cap$ T $\neq \emptyset$ and Z $\cap$ T $\neq \emptyset$.

So there exists $x \in$ X $\cap$T, $y \in$ Y $\cap$T and $z \in$ Z $\cap$T. Now $xyz \in$ XYZ $\cap$T $\subseteq$ M $\cap$T =$\emptyset$. It is a contradiction. Therefore either X $\subseteq$ M or Y $\subseteq$ M or Z $\subseteq$ M and hence M is a prime ideal of T containing A.

Now A $\subseteq$ M $\subseteq$ T \ S $\subseteq$ P. Since P is a minimal prime ideal relative to containing A, we have M = T\S = P and T\P = S. Let $xyz \in$ P. Then $xyz \notin$ S. Suppose if possible $x \notin$ P, $y \notin$ P and $z \notin$ P then $x, y, z \in$ T\P and hence $x, y, z \in$ S $\Rightarrow xyz \in$ T. It is contradiction. Therefore $x \in$ P or $y \in$ P or $z \in$ P. Therefore P is a completely prime ideal of T.

Theorem 3.31: Let P an ideal of a duo semi group T. Then P is completely prime if and only if P is prime.

Proof: Suppose that P is completely semiprime ideal of duo semi group T. By theorem ......, P is a prime ideal of T. Conversely suppose that P is a prime ideal of T. Let $x, y, z \in$ T and $xyz \in$ P. Now P is an ideal of duo ternary semigroup T and $xyz \in$ P, by theorem 3.16, $< x > < y >< z > \subseteq$ P. Since P is prime, $< x > \subseteq$ P or $< y > \subseteq$ P or $< z > \subseteq$ P.

Hence $x \in$ P or $y \in$ P or $z \in$ P. Hence P is a completely prime ideal of S.

Theorem 3.32: If T is a duo ternary semigroup and A is an ideal of S then $A_1 = A_3$.

Proof: By theorem 3.31, in a duo ternary semigroup T, An ideal P is prime iff P is completely prime. Hence $A_1 = A_3$.

Theorem 3.33: If T is a duo ternary semigroup and A is an ideal of T, then $A_1=A_2=A_3=A_4$.

Proof: By theorem 3.32, in a duo ternary semigroup, An ideal P is prime iff P is completely prime. So $A_1 = A_3$. By theorem 3.28, $A_2 = A_4$. By theorem 2.49, $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. Now $A_4 \subseteq A_3 \subseteq A_2$ and $A_4 = A_2$ implies $A_2 = A_3 = A_4$. Also $A_3 \subseteq A_2 \subseteq A_1$ and $A_1 = A_3$ implies $A_3 = A_2 = A_1$. Hence $A_1 = A_2 = A_3 = A_4$.

Definition 3.34: A ternary semigroup T is said to be an ***archimedian ternary semigroup*** if for any $a, b \in$ T, there is an odd natural number *n* such that $a^n \in TbT$.

Definition 3.35: A ternary semigroup T is said to be an ***strongly archimedian ternary semigroup*** if for any $a, b \in$ S, there is an odd natural number *n* such that $<a>^n \subseteq TbT$.

Theorem 3.36: Every strongly archemedian ternary semigroup is an archemedian semi group.

Proof: Suppose that T is a strongly archemedian ternary semigroup. For any $a, b \in$ T, there exists an odd natural number *n* such that $< a >^n \subseteq < b >$.

Therefore $a^{n+2} \in < a >< a >^n < a > \subseteq < a >< b >< a > \subseteq TbT$. Therefore T is archemedian ternary semigroup.

Theorem 3.37: Let T be a duo semi group. Then T is archemedian if and only if T is strongly archemedian.

Proof: Suppose that T is strongly archemedian ternary semigroup. Then by theorem 3.34, T is archemedian. Conversely suppose that T is an archemedian ternary semigroup. Let $a, b, c \in$ T. Since T is archemedian, there exists a natural number n such that $a^n \in TbT$. Since TbT is an ideal of a duo ternary semigroup T, by corollary 3.18, $a^n \in TbT \Rightarrow < a >^n \subseteq TbT$. Therefore T is a strongly archemedian ternary semigroup.

Theorem 3.38: Let T be a duo ternary semigroup. Then T is archemedian if and only if T has no proper prime ideals.

Proof: Suppose that T is an archemedian duo ternary semigroup. Let P be a prime ideal of T. Let $a \in$ P and $b \in$ T. P is prime, we have $TaT \subseteq P$. Since T is archemedian, $b^n \in TaT$ for some odd natural number n. Thus $b^n \in TaT \subseteq P$. Since T is a duo semi group, By theorem 3.31, P is completly prime. Thus $b^n \in P \Rightarrow b \in P$. Hence T = P. Therefore T has no proper prime ideals.

Conversely suppose that T has no proper prime ideals. Let $a, b \in$ T. For any $b \in$ T, the intersection of all prime ideals of T containing B = $< b >$ is T self. Therefore $\sqrt{< b >}$ = T. Now $a \in T = \sqrt{< b >} \Rightarrow a^n \in < b >$ for some odd natural number n. Since T is duo ternary semigroup, By corollary 3.18, $< a >^n \subseteq < b >$ for some odd natural number n.

Therefore $< a >^{n+2} \subseteq < a >< b >< a > \subseteq T< b >T$. Thus T is strongly archemedian. Hence T is archemedian.

Theorem 3.39: If T is a duo ternary semigroup, then S = $\{a \in T : \sqrt{< a >} \neq T \}$ is either empty or prime ideal.

Proof: Suppose that S is nonempty subset of duo ternary semigroup T. Let $a \in$ T and $s, t \in$T.

Now $a \in T \Rightarrow \sqrt{< a >} \neq T$. Therefore $\sqrt{< ast >} \neq T$, $\sqrt{< sta >} \neq T$ and $\sqrt{< sat >} \neq T$. Therefore $ast, sta, sat \in S$.

Therefore S is an ideal of T. Suppose that $abc \in S$ for some $a, b, c \in$T. If possible suppose that $a \notin S$, $b \notin S$ and $c \notin S$.

We have $\sqrt{< a >} = \sqrt{< b >} = \sqrt{< c >}$ = T. Therefore T = $\sqrt{< a >} \cap \sqrt{< b >} \cap \sqrt{< c >} = \sqrt{< a > \cap < b > \cap < c >} = \sqrt{< a >< b >< c >} = \sqrt{< abc >} \neq T$. It is a contradiction. Therefore $a \in S$ or $b \in S$ or $c \in S$. Hence S is a completely prime ideal of T. Since T is duo ternary semigroup, by theorem 3.31, S is a prime ideal of T.

Theorem 3.40: If T is a duo ternary semigroup and S = $\{ a \in T : \sqrt{< a >} \neq T \}$, then T\S is either empty or an archemedian subternary semigroup of T.

Proof: Let $a, b, c \in T \setminus S$. Then $\sqrt{< a >} = \sqrt{< b >} = \sqrt{< c >}$ = T. Therefore $b, c \in \sqrt{< a >}$, $b^n \in < a >$ for some odd natural number n. Therefore $b^{n+2} \in TaT \Rightarrow b^{n+2} = sat$ for some $s, t \in$ S. If either s or $t \in$ S, then $b^{n+2} \in$ S. By theorem 3.39, S is prime and hence $b \in$ S. This is contradiction. Therefore $s, t \in T \setminus S$. Thus $b^{n+2} = sat \in T\backslash S a T\backslash S$. Hence T\S is an archemedian ternary semigroup.

Theorem 3.41: If M is a nontrivial maximal ideal of a duo ternary semigroup T then M is prime.

Proof: Suppose if possible M is not prime. Then there exists $a, b, c \in$ T such that $<a> <b> <c> \subseteq M$ and $a, b, c \notin M$. Now for any $x \in T\backslash M$, we have $T = M \cup <b> = M \cup <c> = M \cup <x>$. Since $b, c, x \in T\backslash M$, we have $b, c \in <x>$ and $x \in <b>$, $x \in <c>$. So $<b> = <c> = <x>$. Therefore $<b>^3 \subseteq M$, $<c>^3 \subseteq M$. If $a \neq b$, then $a = sbt$ for some $s, t \in T^1$. So $a \in <s> <b> <t>$. If either $s \in M$ or $t \in M$ then $a \in M$. It is a contradiction. If $s \notin M$ and $t \notin M$, then $<s> <b><t> \subseteq <b>^3 \subseteq M$. $\therefore a \in <s> <b> <t> \subseteq M$. Therefore $a \in M$. It is a contradiction. Thus $a = b$ and hence M is trivial, which is not true. So M is prime.

Theorem 3.42: If T is a duo ternary semigroup and contains a nontrivial maximal ideal then T contains semisimple elements.

Proof: Let M be a nontrivial maximal ideal of T. By theorem 3.39, M is prime.

Let $a \in T\backslash M$. Then $< a > \not\subseteq M$. Since M is maximal, $M \cup < a > = T$. If $< a >^3 \subseteq M$, then $< a > \subseteq M$ which is not true. So $< a >^3 \not\subseteq M$. Since M is maximal, $M \cup < a >^3 = T$.

Now $M \cup < a > = M \cup < a >^3$. Therefore $a \in < a >^3$ and hence a is semisimple.

Theorem 3.43: If T is a duo ternary semigroup, then the following are equivalent.
1) T is a strongly archimedean ternary semigroup.
2) T is an archimedean ternary semigroup.
3) T has no proper completely prime ideals.
4) T has no proper completely semiprime ideals.
5) T has no proper prime ideals.
6) T has no proper semiprime ideals.

Proof: By theorem 3.37, (1) and (2) are equivalent.
By theorem 3.38, (2) and (5) are equivalent.
By theorem 3.33, (3), (4), (5) and (6) are equivalent.
Hence the given conditions are equivalent.

Theorem 3.44: Let T be a duo archimedian ternary semigroup. Then an ideal M is maximal if and only if M is trivial. Also if $T = T^3$, then T has no maximal ideals.

Proof: If M is a trivial ideal of T then clearly M is a maximal ideal. Suppose that M is maximal. Suppose if possible M is nontrivial. By theorem 3.39, M is prime. Since T is an archimedian duo ternary semigroup, by theorem 3.36, T has no proper prime ideals. It is a contradiction. Therefore M is trivial. Suppose that $T = T^3$. Suppose if possible T has a maximal ideal M. By theorem 3.39, M is prime. Since T is archimedian duo ternary semigroup, by theorem 3.36, T has no proper prime ideals. It is a contradiction. Therefore T has no maximal ideals.

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