



**A STUDY ON THE DEGREE OF APPROXIMATION OF FUNCTION
BY ALMOST MATRIX MEANS OF THE CONJUGATE FOURIER SERIES**

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(Received on: 27-04-13; Revised & Accepted on: 05-06-13)

ABSTRACT

In the present paper, we obtained the degree of approximation of functions belonging to weighted $W(L, \xi(t))$, $r \geq 1$, class by almost infinite regular triangular matrix means of a conjugate Fourier series.

1. INTRODUCTION

Let $f(x)$ be the periodic function with period 2π and integrable in the Lebesgue sense. The Fourier series of $f(x)$ is given by

$$f \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x). \tag{1.1}$$

With partial sums $s_n(f; x)$.

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx + a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

L^∞ norm of a function $f: R \rightarrow R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)|: x \in R\}. \tag{1.3}$$

L_r norm of a function is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right)^{1/r}, r \geq 1. \tag{1.4}$$

The degree of approximation of a function $f: R \rightarrow R$ by a trigonometric polynomial t_n of order n under norm $\|\cdot\|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)|: x \in R\}. \tag{1.5}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min \|t_n - f\|_r. \tag{1.6}$$

A function $f \in Lip \alpha$ if

$$|f(x+t) - f(x)| = O(|t|^\alpha) \text{ for } 0 < \alpha \leq 1 \tag{1.7}$$

$f(x) \in Lip(\alpha, r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{1/r} = O(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1. \tag{1.8}$$

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$ $f \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{1/r} = O(\xi(t)) \tag{1.9}$$

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and that $f \in W(L_r, \xi(t))$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)| \sin^\beta x \, dx \right)^{1/r} = O(\xi(t)), \beta \geq 0, r \geq 1. \quad (1.10)$$

In case $\beta=0$ we find that $W(L_r, \xi(t))$, reduces to the class $Lip(\xi(t), r)$ and if $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class reduces to the class $Lip(\alpha, r)$ and if $r \rightarrow \infty$ then $Lip(\alpha, r)$ class reduces to the class $Lip \alpha$.

We observe that

$$W(L_r, \xi(t)) \xrightarrow{\beta=0} Lip(\xi(t), r) \xrightarrow{\xi(t)=t^\alpha} Lip(\alpha, r) \xrightarrow{r \rightarrow \infty} Lip \alpha.$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n th partial sums $\{s_n\}$, then the bounded sequence $\{S_n\}$ is said to be almost convergent to a limit S (see Lorenz [3]), if

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=p}^{n+p} s_k \rightarrow S \quad (1.11)$$

Uniformly with respect to p .

Let $\Lambda_{n,k} = (a_{n,k})$ be an infinite regular triangular matrix satisfying the Silverman-TÖeplitz [9] condition of regularity, such that

- (1) $\sum_{k=0}^n \Lambda_{n,k} \rightarrow 1$ as $n \rightarrow \infty$.
- (2) $\Lambda_{n,k} = 0$; for all $k > n$.
- (3) $\sum_{k=0}^n |\Lambda_{n,k}| \leq M$ where M is a finite constant.

A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{s_n\}$ is said to be almost Matrix summable to S , Provided.

$$\Lambda_{n,m} = \sum_{k=0}^n \Lambda_{n,k} s_{k,m} \rightarrow S \text{ as } n \rightarrow \infty.$$

Uniformly with respect to m , where

$$s_{k,m} = \frac{1}{k+1} \sum_{v=m}^{k+m} s_v.$$

and $(a_{n,k})$ is an infinite regular triangular matrix such that the element $a_{n,k}$ is non-negative, non-decreasing with k , so that for every

$$\left| \sum_{k=0}^n \Lambda_{n,k} \right| = O(1)$$

We shall use following notations:

1. $\phi(t) = f(x+t) + f(x-t) - 2f(x)$
2. $\bar{\Lambda}_n^t = \frac{1}{2\pi} \left| \sum_{k=0}^n \frac{a_{n,n-k}}{(k+1)} \frac{\cos(k+2m+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 t/2} \right|$

2. PREVIOUS RESULTS

The concept of almost convergence introduced by Lorentz[3] in 1948. Till now the concept of almost convergence have been studied by several researchers like Mazhar, S.M. and Siddiqui[4], Qureshi[6] and Lal[2] etc. The purpose of this paper is to obtain a new theorem on the degree of approximation using the concept of almost matrix mean. Our object of this paper is to prove the following:

3. MAIN THEOREM

Let $\Lambda_{n,m} = (a_{n,m})$ be an infinite regular triangular matrix such that the element $(a_{n,m})$ is non-negative, non-decreasing with k , so that for every n ,

$$|\sum_{m=0}^n \Lambda_{n,m}| = O(1) \tag{3.1}$$

If a function \bar{f} , conjugate to a 2π -periodic function f on $[0, 2\pi]$ and belongs to the weighted $W(L_r, \xi(t))$ class, $r \geq 1$, then its degree of approximation by almost infinite regular triangular matrix means of its conjugate Fourier series (1.2) is given by

$$\|\bar{\Lambda}_{n,m}(x) - \bar{f}(x)\|_r = O\left((n+m)^{\beta + \frac{1}{r}} \xi\left(\frac{1}{n+m}\right)\right). \tag{3.2}$$

Provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{3.3}$$

$$\left[\int_0^{1/n+m} \left\{\frac{t|\phi(t)|}{\xi(t)} \sin^\beta t\right\}^r dt\right]^{1/r} = O\left(\frac{1}{n+m}\right), \tag{3.4}$$

and

$$\left[\int_{1/n+m}^\pi \left\{\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \sin^\beta t\right\}^r dt\right]^{1/r} = O((n+m)^\delta) \tag{3.5}$$

where δ is an arbitrary number such that $(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, $1 \leq r \leq \infty$, both (3.4) and (3.5) conditions hold uniformly in x , and $\bar{\Lambda}_{n,m}$ is almost infinite regular triangular matrix means of the conjugate Fourier series (1.2).

4. LEMMAS

For the proof of the theorem, we shall require the following lemma:

Lemma 4.1: For $0 \leq t \leq \frac{1}{n+m}$; $|\bar{\Lambda}_n^t| = O\left(\frac{1}{t}\right)$.

Proof: We have,

$$\bar{\Lambda}_n^t = \frac{1}{2\pi} \left| \sum_{k=0}^n \frac{a_{n,n-k}}{(k+1)} \frac{\cos(k+2m+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 t/2} \right|$$

Since $|\cos nt| \leq 1$ and $\sin t/2 \geq t/\pi$, for $0 \leq t \leq \frac{1}{n+m}$

$$\begin{aligned} &\leq \left| \frac{1}{2\pi} \sum_{k=0}^n \frac{a_{n,n-k}}{(k+1)} \left\{ (k+1) \frac{1}{\pi} \right\} \right| \\ &= \frac{1}{2t} \left| \sum_{k=0}^n a_{n,n-k} \right| \\ &= O\left(\frac{1}{t}\right) \quad (\text{since } |\sum_{k=0}^n a_{n,n-k}| = O(1)) \end{aligned}$$

Lemma 4.2: For $\frac{1}{n+m} \leq t \leq \pi$; $|\bar{\Lambda}_n^t| = O\left(\frac{1}{t}\right)$.

Proof: We have,

$$\bar{\Lambda}_n^t = \frac{1}{2\pi} \left| \sum_{k=0}^n \frac{a_{n,n-k}}{(k+1)} \frac{\cos(k+2m+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 t/2} \right|$$

Using Jordan's lemma, $\sin \frac{t}{2} \geq \frac{t}{\pi}$, $\sin nt \leq 1$ and $|\cos nt| \leq 1$, we have

$$\begin{aligned} &\leq \frac{1}{2\pi} \left| \sum_{k=0}^n \frac{a_{n,n-k}}{(k+1)} \right| \left\{ \frac{\left| \cos(2k+1) \frac{t}{2} \right| \left| \sin(k+1) \frac{t}{2} \right|}{\left| \sin^2 \frac{t}{2} \right|} \right\} \\ &= 0 \left[\frac{1}{2t} \left| \sum_{k=0}^n a_{n,n-k} \right| \right] \\ &= 0 \left(\frac{1}{t} \right) \quad ; \left(\text{since } \left| \sum_{k=0}^n a_{n,n-k} \right| = O(1) \right) \end{aligned}$$

4. PROOF OF THE MAIN THEOREM

Let $\bar{s}_n(f; x)$ denote the n^{th} partial sum of conjugate Fourier series (1.2) at $t = x$, then the following Qureshi[6], we have

$$\bar{s}_n(f; x) - \bar{f}(x) = -\frac{1}{\pi} \int_0^\pi \phi(t) \frac{\cos(n+1/2)t}{2\sin t/2} dt$$

Therefore,

$$\begin{aligned} \bar{s}_{k,m}(f; x) - \bar{f}(x) &= \frac{1}{k+1} \sum_{v=m}^{k+m} \{ \bar{s}_v(f; x) - \bar{f}(x) \} \\ &= -\frac{1}{\pi(k+1)} \int_0^\pi \phi(t) \sum_{v=m}^{k+m} \frac{\cos(v+1/2)t}{2\sin t/2} dt \\ &= \frac{1}{2\pi(k+1)} \int_0^\pi \phi(t) \frac{\sin kt - \sin(k+1)t}{2\sin^2 t/2} dt \end{aligned}$$

Therefore using, (1.2) the almost infinite regular triangular matrix transform of $\bar{s}_n(f; x)$ is given by

$$\begin{aligned} \bar{\Lambda}_{n,m}(x) - \bar{f}(x) &= \sum_{k=0}^n a_{n,k} \{ \bar{s}_{k,m}(f; x) - \bar{f}(x) \} \\ \bar{\Lambda}_{n,m}(x) - \bar{f}(x) &= \frac{1}{2\pi} \int_0^\pi \phi(t) \sum_{k=0}^n \frac{a_{n,n-k}}{(k+1)} \frac{\sin kt - \sin(k+1)t}{2\sin^2 t/2} dt \end{aligned}$$

Therefore

$$\begin{aligned} |\bar{\Lambda}_{n,m}(x) - \bar{f}(x)| &\leq \frac{1}{2\pi} \int_0^\pi |\phi(t)| \left| \sum_{k=0}^n \frac{a_{n,n-k}}{(k+1)} \frac{\cos(k+2m+1) \frac{t}{2} \sin(k+1) \frac{t}{2}}{\sin^2 t/2} \right| dt \\ &= \int_0^\pi |\phi(t)| |\bar{\Lambda}_n^t| dt \\ &= \int_0^{\frac{1}{n+m}} |\phi(t)| |\bar{\Lambda}_n^t| dt + \int_{\frac{1}{n+m}}^\pi |\phi(t)| |\bar{\Lambda}_n^t| dt \\ &= I_1 + I_2 \quad , (\text{say}) \end{aligned} \tag{5.1}$$

Let us consider I_1 first,

$$|I_1| = \int_0^{\frac{1}{n+m}} |\phi(t)| |\bar{\Lambda}_n^t| dt$$

Using Hölder's inequality and in view of $(\sin t)^{-1} \leq \frac{\pi}{t}$, $\sin t \geq \frac{2t}{\pi}$, we have

$$\begin{aligned}
 &= \left[\int_0^{\frac{1}{n+m}} \left\{ \frac{t|\phi(t)|}{\xi(t)} \sin^\beta t \right\}^r dt \right]^{1/r} \left[\int_0^{\frac{1}{n+m}} \left\{ \frac{\xi(t)|\bar{\Lambda}_n^t|}{t \sin^\beta t} \right\}^s dt \right]^{1/s} \\
 &= O\left(\frac{1}{n+m}\right) \left[\int_0^{\frac{1}{n+m}} \left\{ \frac{\xi(t)}{t^{1+1+\beta}} \right\}^s dt \right]^{1/s} \quad (\text{By Lemma 4.1})
 \end{aligned}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem, for integrals, for $0 < \epsilon < \frac{1}{n+1}$

$$\begin{aligned}
 &= O\left\{ \left(\frac{1}{n+m}\right) \xi\left(\frac{1}{n+m}\right) \right\} \left[\int_\epsilon^{\frac{1}{n+m}} t^{-(\beta+2)s} dt \right]^{1/s} \\
 &= O\left\{ \xi\left(\frac{1}{n+m}\right) \left[\frac{t^{-(\beta+2)+1/s}}{-(\beta+2)+1/s} \right]_\epsilon^{\frac{1}{n+m}} \right\} \\
 &= O\left\{ \xi\left(\frac{1}{n+m}\right) (n+m)^{\beta+1-1/s} \right\} \quad \left(\text{since } \frac{1}{r} + \frac{1}{s} = 1 \right) \\
 &= O\left\{ (n+m)^{\beta+1/r} \xi\left(\frac{1}{n+m}\right) \right\} \quad (5.2)
 \end{aligned}$$

Now consider I_2 ,

$$|I_2| = \int_{\frac{1}{n+m}}^{\pi} |\phi(t)| |\bar{\Lambda}_n^t| dt$$

Again, using Hölder's inequality and second mean value theorem for integrals, since $|\sin t| \leq 1$, $\sin t \geq \frac{2t}{\pi}$, with conditions (2.2) and (2.4), we have

$$\begin{aligned}
 |I_2| &= \left[\int_{\frac{1}{n+m}}^{\pi} \left\{ \frac{t^{-\delta} |\phi(t)|}{\xi(t)} \sin^\beta t \right\}^r dt \right]^{1/r} \left[\int_{\frac{1}{n+m}}^{\pi} \left\{ \frac{\xi(t) |\bar{\Lambda}_n^t|}{t^{-\delta} \sin^\beta t} \right\}^s dt \right]^{1/s} \\
 &= O\{(n+m)^\delta\} \left[\int_{\frac{1}{n+m}}^{\pi} \left\{ \frac{\xi(t)}{t^{1-\delta+\beta}} \right\}^s dt \right]^{1/s} \quad (\text{by lemma 4.2})
 \end{aligned}$$

Now put $(t = \frac{1}{y})$

$$\begin{aligned}
 &= O\{(n+m)^\delta\} \left[\int_{\frac{1}{\pi}}^{n+m} \left\{ \frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1}} \right\}^s \frac{dy}{y^2} \right]^{1/s} \\
 &= O\left\{ (n+m)^\delta \xi\left(\frac{1}{n+m}\right) \right\} \left[\int_1^{n+m} \left\{ \frac{dy}{y^{s(\delta-\beta-1)+2}} \right\} \right]^{1/s} \quad \left(\text{for } \frac{1}{\pi} \leq 1 \leq (n+m) \right)
 \end{aligned}$$

$$\begin{aligned}
 &= O \left\{ (n+m)^\delta \xi \left(\frac{1}{n+m} \right) \left[\left\{ \frac{y^{s(1+\beta-\delta)-1}}{s(1+\beta-\delta)-1} \right\}_1^{n+m} \right]^{1/s} \right\} \\
 &= O \left\{ (n+m)^\delta \xi \left(\frac{1}{n+m} \right) \left[(n+m)^{1+\beta-\delta-\frac{1}{s}} \right] \right\} \\
 &= O \left\{ \xi \left(\frac{1}{n+m} \right) (n+m)^{\beta+1-\frac{1}{s}} \right\} \\
 &= O \left\{ (n+m)^{\beta+\frac{1}{r}} \xi \left(\frac{1}{n+m} \right) \right\}; \left(\text{since } \frac{1}{r} + \frac{1}{s} = 1 \right) \tag{5.3}
 \end{aligned}$$

Combining (5.1), (5.2) and (5.3), we get

$$| \bar{\Lambda}_{n,m}(x) - \bar{f}(x) | = O \left\{ (n+m)^{\beta+\frac{1}{r}} \xi \left(\frac{1}{n+m} \right) \right\}$$

Now, using L_r -norm, we get

$$\begin{aligned}
 \| \bar{\Lambda}_{n,m}(x) - \bar{f}(x) \|_r &= \left\{ \int_0^{2\pi} | \bar{\Lambda}_{n,m}(x) - \bar{f}(x) |^r dx \right\}^{\frac{1}{r}} \\
 &= O \left\{ (n+m)^{\beta+\frac{1}{r}} \xi \left(\frac{1}{n+m} \right) \left\{ \int_0^{2\pi} dx \right\}^{\frac{1}{r}} \right\} \\
 &= O \left\{ (n+m)^{\beta+\frac{1}{r}} \xi \left(\frac{1}{n+m} \right) \right\}
 \end{aligned}$$

This completes the proof of the theorem.

6. COROLLARIES

Corollary 6.1: If In case $\beta=0$ and $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$, $W(L_r, \xi(t))$, reduces to the class $Lip(\xi(t), r)$, and the degree of approximation of a function $f \in Lip(\alpha, r)$, $\frac{1}{r} < \alpha < 1$ is given by

$$\| \bar{\Lambda}_{n,m}(x) - \bar{f}(x) \|_r = O \left\{ \frac{1}{(n+m)^{\alpha-\frac{1}{r}}} \right\}.$$

Corollary 6.2: If $r \rightarrow \infty$, in, Corollary 6.1, then, $Lip(\alpha, r)$, reduces to the class $Lip \alpha$, and the degree of approximation of a function $f \in Lip \alpha$, for $0 < \alpha < 1$ is given by

$$\| \bar{\Lambda}_{n,m}(x) - \bar{f}(x) \|_\infty = O \left\{ \frac{1}{(n+m)^\alpha} \right\}.$$

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Source of support: Nil, Conflict of interest: None Declared