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A STUDY ON THE DEGREE OF APPROXIMATION OF FUNCTION BY ALMOST MATRIX MEANS OF THE CONJUGATE FOURIER SERIES

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ABSTRACT

In the present paper, we obtained the degree of approximation of functions belonging to weighted $W(L^r, \xi(t)), r \geq 1$, class by almost infinite regular triangular matrix means of a conjugate Fourier series.

1. INTRODUCTION

Let $f(x)$ be the periodic function with period 2π and integrable in the Lebesgue sense. The Fourier series of f(x) is given by

$$f \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x). \tag{1.1}$$

With partial sums $s_n(f; x)$.

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx + a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

$L\infty$ -norm of a function $f: R \to R$ is defined by

$$\| f\|_\infty = sup\{|f(x)|: x \epsilon R\}. \tag{1.3}$$

L_r = norm of a function is defined by

$$\|f\|_r = \left(\int_0^{2\pi}|f(x)|^r dx\right)^{1/r}, r \geq 1. \tag{1.4}$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial t_n of order n under norm $\| . \|_\infty$ is defined by

$$\| t_n - f\|_\infty = sup\{|t_n(x) - f(x)|: x \epsilon R\}. \tag{1.5}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = min\|t_n - f\|_r. \tag{1.6}$$

A function $f \in Lip\, \alpha$ if

$$|f(x+t) - f(x)| = O(|t|^\alpha) for\ 0 < \alpha \leq 1 \tag{1.7}$$

$f(x) \in Lip(\alpha, r)$, if

$$\left(\int_0^{2\pi}|f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1. \tag{1.8}$$

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$ $f \in$ Lip $(\xi(t), r)$ if

$$\left(\int_0^{2\pi}|f(x+t) - f(x)|^r dx\right)^{1/r} = O(\xi(t)) \tag{1.9}$$

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and that $f\epsilon\ W\big(L_r, \xi(t)\big)$, if

$$\left(\int_0^{2\pi}\left|\{f(x+t)-f(x)\}sin^{\beta}x\right|^r dx\right)^{1/r} = O(\xi(t)), \beta \geq 0\ r \geq 1. \tag{1.10}$$

In case β=0 we find that $W\big(L_r, \xi(t)\big)$, reduces to the class Lip $(\xi(t),\ r)$ and if $\xi(t) = t^{\alpha}$ then Lip $(\xi(t),\ r)$ class reduces to the class *Lip* (α , *r*) and if $r \to \infty$ then *Lip* (α , *r*) class reduces to the class *Lip* α.

We observe that

$$W\big(L_r, \xi(t)\big) \xrightarrow{\beta=0} \text{Lip}\ (\xi(t), r) \xrightarrow{\xi(t)=t^{\alpha}} \text{Lip}\ (\alpha,\ r) \xrightarrow{r\to\infty} \text{Lip}\alpha.$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its *nth* partial sums $\{s_n\}$, then the bounded sequence $\{S_n\}$ is said to be almost convergent to a limit S (see Lorenz [3]), if

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=p}^{n+p} s_k \to S \tag{1.11}$$

Uniformly with respect to p.

Let $\Lambda_{n,k} = \big(a_{n,k}\big)$ be an infinite regular triangular matrix satisfying the Silverman-TÖeplitz [9] condition of regularity, such that

(1) $\sum_{k=0}^{n} \Lambda_{n,k} \to 1$ as $n \to \infty$.

(2) $\Lambda_{n,k} = 0$; for all $k > n$.

(3) $\sum_{k=0}^{n} \left|\Lambda_{n,k}\right| \leq M$ where M is a finite constant.

A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums$\{s_n\}$ is said to be almost Matrix summable to S, Provided.

$$\Lambda_{n,m} = \sum_{k=0}^{n} \Lambda_{n,k}\, s_{k,m} \to S \text{ as } n \to \infty.$$

Uniformly with respect to m, where

$$s_{k,m} = \frac{1}{k+1}\sum_{v=m}^{k+m} s_v.$$

and $\big(a_{n,k}\big)$ is an infinite regular triangular matrix such that the element $a_{n,k}$ is non-negative, non-decreasing with k, so that for every n,

$$\left|\sum_{k=0}^{n} \Lambda_{n,k}\right| = O(1)$$

We shall use following notations:

1. $\phi(t) = f(x+t) + f(x-t) - 2f(x)$
2. $\overline{\Lambda}_n^t = \frac{1}{2\pi}\left|\sum_{k=0}^{n} \frac{a_{n,n-k}}{(k+1)} \frac{\cos{(k+2m+1)}\frac{t}{2} sin{(k+1)}\frac{t}{2}}{sin^2 t/2}\right|$

2. PREVIOUS RESULTS

The concept of almost convergence introduced by Lorentz[3] in 1948. Till now the concept of almost convergence have been studied by several researchers like Mazhar, S.M. and Siddiqui[4], Qureshi[6] and Lal[2] etc. The purpose of this paper is to obtain a new theorem on the degree of approximation using the concept of almost matrix mean. Our object of this paper is to prove the following:

3. MAIN THEOREM

Let $\Lambda_{n,m} = \left(a_{n,m}\right)$ bean infinite regular triangular matrix such that the element $\left(a_{n,m}\right)$ is non-negative, non-decreasing with k, so that for every n,

$$\left|\sum_{m=0}^{n} \Lambda_{n,m}\right| = O(1) \tag{3.1}$$

If a function $\bar{f}$, conjugate to a 2π-periodic function *f* on $[0,2\pi]$ and belongs to the weighted $W\left(L_r, \xi(t)\right)$ class,$r \geq 1$, then its degree of approximation by almost infinite regular triangular matrix means of its conjugate Fourier series(1.2) is given by

$$\left\| \bar{\Lambda}_{n,m}(x) - \bar{f}(x)\right\|_r = O\left((n+m)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+m}\right)\right). \tag{3.2}$$

Provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} be\ a\ decreasing\ sequence, \tag{3.3}$$

$$\left[\int_0^{1/n+m} \left\{\frac{t|\phi(t)|}{\xi(t)} sin^{\beta} t\right\}^r dt\right]^{1/r} = O\left(\frac{1}{n+m}\right), \tag{3.4}$$

and

$$\left[\int_{1/n+m}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)|}{\xi(t)} sin^{\beta} t\right\}^r dt\right]^{1/r} = O\left((n+m)^{\delta}\right) \tag{3.5}$$

where δ is an arbitrary number such that$s(1-\delta)-1>0, \frac{1}{r}+\frac{1}{s}=1, 1 \leq r \leq \infty$, both $(3.4) and (3.5)$ conditions hold uniformly in x, and $\bar{\Lambda}_{n,m}$ is almost infinite regular triangular matrix means of the conjugate Fourier series (1.2).

4. LEMMAS

For the proof of the theorem, we shall require the fallowing lemma:

Lemma 4.1: For$0 \leq t \leq \frac{1}{n+m}$; $|\bar{\Lambda}_n^t| = O\left(\frac{1}{t}\right)$.

Proof: We have,

$$\bar{\Lambda}_n^t = \frac{1}{2\pi}\left|\sum_{k=0}^{n} \frac{a_{n,n-k}}{(k+1)} \frac{\cos(k+2m+1)\frac{t}{2} sin(k+1)\frac{t}{2}}{sin^2 t/2}\right|$$

Since$|cosnt| \leq 1$ and $\sin {t}/{2} \geq {t}/{\pi}$, for $0 \leq t \leq \frac{1}{n+m}$

$$\leq \left|\frac{1}{2\pi}\sum_{k=0}^{n} \frac{a_{n,n-k}}{(k+1)}\left\{(k+1)\frac{1}{\frac{t}{\pi}}\right\}\right|$$

$$= \frac{1}{2t}\left|\sum_{k=0}^{n} a_{n,n-k}\right|$$

$$= O\left(\frac{1}{t}\right) \qquad \left(since \left|\sum_{k=0}^{n} a_{n,n-k}\right| = O(1)\right)$$

Lemma 4.2: For$\frac{1}{n+m} \leq t \leq \pi$; $|\bar{\Lambda}_n^t| = O\left(\frac{1}{t}\right)$..

Proof: We have,

$$\bar{\Lambda}_n^t = \frac{1}{2\pi}\left|\sum_{k=0}^{n} \frac{a_{n,n-k}}{(k+1)} \frac{\cos(k+2m+1)\frac{t}{2} sin(k+1)\frac{t}{2}}{sin^2 t/2}\right|$$

Using Jordan's lemma, $\sin {}^{t}/_{2} \geq {}^{t}/_{\pi}$, $sinnt \leq 1$ and $|cosnt| \leq 1$, we have

$$\leq \frac{1}{2\pi}\left|\sum_{k=0}^{n}\frac{a_{n,n-k}}{(k+1)}\right|\left\{\frac{\left|\cos(2k+1)\frac{t}{2}\right|\left|sin(k+1)\frac{t}{2}\right|}{\left|sin^2\frac{t}{2}\right|}\right\}$$

$$= O\left[\frac{1}{2t}\left|\sum_{k=0}^{n}a_{n,n-k}\right|\right]$$

$$= O\left(\frac{1}{t}\right) \qquad ;\left(since\ \left|\sum_{k=0}^{n}a_{n,n-k}\right| = O(1)\right)$$

4. PROOF OF THE MAIN THEOREM

Let $\overline{s_n}(f;x)$ denote the n[th] partial sum of conjugate Fourier series (1.2) at $t = x$, then the following Qureshi[6], we have

$$\overline{s_n}(f;x) - \bar{f}(x) = -\frac{1}{\pi}\int_0^{\pi}\phi(t)\frac{cos(n+1/2)t}{2sint/2}dt$$

Therefore,

$$\bar{s}_{k,m}(f;x) - \bar{f}(x) = \frac{1}{k+1}\sum_{v=m}^{k+m}\{\bar{s}_v(f;x) - \bar{f}(x)\}$$

$$= -\frac{1}{\pi(k+1)}\int_0^{\pi}\phi(t)\sum_{v=m}^{k+m}\frac{cos\,(v+1/2)t}{2sint\,/2}dt$$

$$= \frac{1}{2\pi(k+1)}\int_0^{\pi}\phi(t)\frac{sinkt - sin\,(k+1)t}{2sin^2t/2}dt$$

Therefore using, (1.2) the almost infinite regular triangular matrix transform of $\overline{s_n}(f;x)$ is given by

$$\overline{\Lambda}_{n,m}(x) - \bar{f}(x) = \sum_{k=0}^{n}a_{n,k}\{\bar{s}_{k,m}(f;x) - \bar{f}(x)\}$$

$$\overline{\Lambda}_{n,m}(x) - \bar{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\frac{a_{n,n-k}}{(k+1)}\frac{sinkt - sin(k+1)t}{2sin^2t/2}dt$$

Therefore

$$\left|\overline{\Lambda}_{n,m}(x) - \bar{f}(x)\right| \leq \frac{1}{2\pi}\int_0^{\pi}|\phi(t)|\left|\sum_{k=0}^{n}\frac{a_{n,n-k}}{(k+1)}\frac{\cos(k+2m+1)\frac{t}{2}sin(k+1)\frac{t}{2}}{sin^2t/2}\right|dt$$

$$= \int_0^{\pi}|\phi(t)||\overline{\Lambda}_n^t|\,dt$$

$$= \int_0^{\frac{1}{n+m}}|\phi(t)||\overline{\Lambda}_n^t|dt + \int_{\frac{1}{n+m}}^{\pi}|\phi(t)||\overline{\Lambda}_n^t|dt$$

$$= I_1 + I_2 \quad ,(say) \tag{5.1}$$

Let us consider I_1 first,

$$|I_1| = \int_0^{\frac{1}{n+m}}|\phi(t)||\overline{\Lambda}_n^t|dt$$

Using Hölder's inequality and in view of $(sint)^{-1} \leq \frac{\pi}{t},\ sint \geq \frac{2t}{\pi}, we$ have

$$= \left[\int_0^{\frac{1}{n+m}} \left\{\frac{t|\phi(t)|}{\xi(t)} sin^{\beta} t\right\}^r dt\right]^{1/r} \left[\int_0^{\frac{1}{n+m}} \left\{\frac{\xi(t)|\overline{\Lambda}_n^t|}{tsin^{\beta} t}\right\}^s dt\right]^{1/s}$$

$$= O\left(\frac{1}{n+m}\right)\left[\int_0^{\frac{1}{n+m}} \left\{\frac{\xi(t)}{t^{1+1+\beta}}\right\}^s dt\right]^{1/s} \qquad (By\ Lemma\ 4.1)$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem, for integrals, for $0 < \epsilon < \frac{1}{n+1}$

$$= O\left\{\left(\frac{1}{n+m}\right)\xi\left(\frac{1}{n+m}\right)\right\}\left[\int_{\epsilon}^{\frac{1}{n+m}} t^{-(\beta+2)s} dt\right]^{1/s}$$

$$= O\left(\xi\left(\frac{1}{n+m}\right)\right)\left[\frac{t^{-(\beta+2)+1/s}}{t^{-(\beta+2)+1/s}}\right]_{\epsilon}^{\frac{1}{n+m}}$$

$$= O\left\{\xi\left(\frac{1}{n+m}\right)(n+m)^{\beta+1-1/s}\right\} \qquad \left(since\ \frac{1}{r}+\frac{1}{s}=1\right)$$

$$= O\left\{(n+m)^{\beta+1/r}\xi\left(\frac{1}{n+m}\right)\right\} \tag{5.2}$$

Now considerI_2,

$$|I_2| = \int_{\frac{1}{n+m}}^{\pi} |\phi(t)||\overline{\Lambda}_n^t| dt\ dt$$

Again, using Hölder's inequality and second mean value theorem for integrals, since $|sint| \leq 1,\ sint \geq \frac{2t}{\pi}$, with conditions (2.2) and (2.4), we have

$$|I_2| = \left[\int_{\frac{1}{n+m}}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)|}{\xi(t)} sin^{\beta} t\right\}^r dt\right]^{1/r} \left[\int_{\frac{1}{n+m}}^{\pi} \left\{\frac{\xi(t)|\overline{\Lambda}_n^t|}{t^{-\delta} sin^{\beta} t}\right\}^s dt\right]^{1/s}$$

$$= O\{(n+m)^{\delta}\}\left[\int_{1/n+m}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^s dt\right]^{1/s} \qquad (by\ lemma 4.2)$$

Now put $\left(t = \frac{1}{y}\right)$

$$= O\{(n+m)^{\delta}\}\left[\int_{1/\pi}^{n+m} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1}}\right\}^s \frac{dy}{y^2}\right]^{1/s}$$

$$= O\left\{(n+m)^{\delta}\xi\left(\frac{1}{n+m}\right)\right\}\left[\int_1^{n+m} \left\{\frac{dy}{y^{s(\delta-\beta-1)+2}}\right\}\right]^{1/s} \left(for\ \frac{1}{\pi} \leq 1 \leq (n+m)\right)$$

$$= O\left\{(n+m)^{\delta}\xi\left(\frac{1}{n+m}\right)\right\}\left[\left\{\frac{y^{s(1+\beta-\delta)-1}}{s(1+\beta-\delta)-1}\right\}_{1}^{n+m}\right]^{1/s}$$

$$= O\left\{(n+m)^{\delta}\xi\left(\frac{1}{n+m}\right)\right\}\left[(n+m)^{1+\beta-\delta-\frac{1}{s}}\right]$$

$$= O\left\{\xi\left(\frac{1}{n+m}\right)(n+m)^{\beta+1-\frac{1}{s}}\right\}$$

$$= O\left\{(n+m)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+m}\right)\right\};\ \left(since\,\frac{1}{r}+\frac{1}{s}=1\right) \tag{5.3}$$

Combining (5.1), (5.2) and (5.3), we get

$$|\,\overline{\Lambda}_{n,m}(x)-\bar{f}(x)\,| = O\left\{(n+m)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+m}\right)\right\}$$

Now, using L_r-norm, we get

$$\|\,\overline{\Lambda}_{n,m}(x)-\bar{f}(x)\|_r = \left\{\int_0^{2\pi}|\,\overline{\Lambda}_{n,m}(x)-\bar{f}(x)|^r\,dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+m)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+m}\right)\right\}\left\{\int_0^{2\pi}dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+m)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+m}\right)\right\}$$

This completes the proof of the theorem.

6. COROLLARIES

Corollary 6.1: If In case β=0 and $\xi(t)=t^{\alpha}$, $0<\alpha\le 1$, $W(L_r,\xi(t))$, reduces to the class Lip($\xi(t)$, r), and the degree of approximation of a function $f\ \epsilon$ Lip(α,r), $\frac{1}{r}<\alpha<1$ is given by

$$\|\,\overline{\Lambda}_{n,m}(x)-\bar{f}(x)\|_r = O\left\{\frac{1}{(n+m)^{\alpha-\frac{1}{r}}}\right\}.$$

Corollary 6.2: If $r\to\infty$, in, Corollary 6.1, then, Lip(α,r), reduces to the class *Lip* α, and the degree of approximation of a function $f\ \epsilon\ Lip$ α, for $0<\alpha<1$ is given by

$$\|\,\overline{\Lambda}_{n,m}(x)-\bar{f}(x)\|_{\infty} = O\left\{\frac{1}{(n+m)^{\alpha}}\right\}.$$

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