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A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS ON A FUZZY METRIC SPACE CONTROLLED BELOW BY SB-FUNCTIONS INVOLVING RATIONAL PARAMETERS

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ABSTRACT

In this paper we use a special class of function called SB-functions due to Sastry et.al [6], use them with rational terms as control function to obtain a fixed point theorem for four self maps on a fuzzy metric. We observe that the result of Reena Jain et.al [5] follows as a corollary of this result.

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1. INTRODUCTION

Reena Jain et.al [5] proved a common fixed point theorem for four selfmaps on a fuzzy metric space satisfying an inequality containing rational functions. We use SB-functions due to Sastry *et.al* [6] as a control functions involving rational parameters to obtain a common fixed point theorem for four selfmaps on a fuzzy metric space and obtain the result of Reena Jain *et.al* [5] as a corollary.

We start with some definitions.

Definition 1.1: (Zadeh. L.A [8]) A fuzzy set A in a nonempty set X is a function with domain X and values in $[0,1]$.

Definition 1.2: (Schweizer.B and Sklar. A [7]) A function $*: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $*$ satisfies the following conditions:

For $a, b, c, d \in [0,1]$

(i) $*$ is commutative and associative

(ii) $*$ is continuous

(iii) $a * 1 = a \ \forall a \in [0,1]$

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$

Definition 1.3: (Kramosil. I and Michelek. J [4]) A triple $(X, M, *)$ is said to be a fuzzy metric space (FM space, briefly) if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfies the following conditions:

For $x, y, z \in X$ and $s, t > 0$.

(i) $M(x, y, t) > 0, M(x, y, 0) = 0$

(ii) $M(x, y, t) = 1 \ \forall \ t > 0$ if and only if $x = y$

(iii) $M(x, y, t) = M(y, x, t)$

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(v) $M(x, y, \cdot): [0, \infty) \longrightarrow [0,1]$ is continuous.

Then M is called a fuzzy metric space on X.

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The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Definition 1.4: (**George. A and Veeramani. P [1]**) Let $(X, M,*)$ be a fuzzy metric space. Then,

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1 \ \forall\, t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M\big(x_{n+p}, x_n, t\big) = 1 \ \forall\, t > 0$ and $p = 1,2, \ldots$

(iii) An FM –space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.5: Let *X* be a nonempty set and *f* and *g* be selfmaps on *X*. A point *x* in *X* is called a coincidence point of *f* and *g* if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of *f* and *g*.

Definition 1.6: (Jungck. G [2]) Two selfmaps S and T of a fuzzy metric space $(X, M,*)$ are said to be weakly compatible pair if they commute at coincidence points, that is if $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

Definition 1.7: Two selfmaps f and g of a nonempty set *X* are occasionally weakly compatible (OWC), if there is a point *x* in *X* which is a coincidence point of f and g at which f and g commute. That is f and g are occasionally weakly compatible if there exists $x \in X \ni fx = gx$ and $fgx = gfx$.

A pair S and T of maps may be OWC but may not be weakly compatible.

Example 1.8: Let ℝ be the real line with usual metric.

Define $S, T: \mathbb{R} \to \mathbb{R}$ by $Sx = 2x$ and $Tx = x^2$ for all $x \in \mathbb{R}$.

Then 0 and 2 are coincidence points of S and T.

Also $ST0 = 0, TS0 = 0$ but $ST2 = 8 \neq 16 = TS2$.

Therefore S and T are occasionally weakly compatible selfmaps but are not weakly compatible.

Lemma 1.9: (Jungck. G and Rhoades.B.E [3]) Let *X* be a nonempty set and *f* and *g* are OWC selfmaps on *X*. If *f* and *g* have unique point of coincidence, $w = fx = gx$, then *w* is the unique common fixed point of *f* and *g*.

Reena Jain, M.S.Rathore and Naval Singh [5] proved the following Theorem.

Theorem 1.10: (Reena Jain, M.S.Rathore and Naval Singh [5], Theorem 3.1)

Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of *X*. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq min \begin{Bmatrix} M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, t), \\ M(Ax, Ty, t)\left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right) \end{Bmatrix} \tag{1.10.1}$$

for $x, y \in X$ and for $t > 0$.

Then there exists a unique point $w \in X$ such that $w = Aw = Sw$ and a unique point $z \in X$ such that $z = Bz = Tz$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

The following Theorem follows immediately from Theorem 1.10

Theorem 1.11: (Reena Jain, M.S.Rathore and Naval Singh [5], Theorem 3.3)

Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of *X*. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq \phi\left(min \begin{Bmatrix} M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, t), \\ M(Ax, Ty, t)\left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right) \end{Bmatrix} \right)$$

for $x, y \in X$ and for $t > 0$, where $\phi: [0,1] \to [0,1]$ is such that $\phi(t) > t$ for $t \in (0,1)$.
Then there exists a unique common fixed point of A, B, S and T.

2. MAIN RESULTS

In this section, we use a special class Ψ of self maps on $[0,1]$ due to [6], called SB-function and use the members of this class as control functions, to prove a common fixed point theorem (Theorem 2.3) for four self maps on a fuzzy metric space. We also observe that condition (1.10.1) is not symmetric; Hence we restate the theorem, making the condition symmetric, and obtain this result as a corollary of our main result.
We start with

Definition 2.1: (K.P.R.Sastry, G.A.Naidu, D.Narayana Rao, and R.Venkata Bhaskar [6]) A function $\psi: [0,1] \to [0,1]$ is said to be a SB- function if (i) $\psi(t) < t$ and (ii) $\psi^n(t) \to 1$ as $t \to 1$ and $n \to \infty$.

Notation: Let $\Psi = \{\psi/\psi: [0,1] \to [0,1] \; is \; a \; SB - function\}$.

It is shown in [6] that the class Ψ is non-empty.

It may be observed that symmetric property in inequality (1.10.1) of Theorem 1.10 is lacking. Hence we restate Theorem 1.10 as follows.

Theorem 2.2: Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq min\left\{\begin{matrix} M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ M(By, Sx, t)\left(\frac{1+M(By,Ty,t)}{1+M(Ax,Sx,t)}\right), M(Ax, Ty, t)\left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right) \end{matrix}\right\} \tag{2.2.1}$$

for $x, y \in X$ and for $t > 0$.

Then there exists a unique point $w \in X$ such that $w = Aw = Sw$ and a unique point $z \in X$ such that $z = Bz = Tz$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

Now, we state and prove our main result.

Theorem 2.3: Let $(X, M,*)$ be a fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $\psi \in \Psi$ and $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq \psi\left(min\left\{\begin{matrix} M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ M(By, Sx, t)\left(\frac{1+M(By,Ty,t)}{1+M(Ax,Sx,t)}\right), M(Ax, Ty, t)\left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right) \end{matrix}\right\} \right) \tag{2.3.1}$$

for $x, y \in X$ and for $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Proof: Since (A, S) and (B, T) are OWC self maps on X, there exist points $x, y \in X$ such that $Ax = Sx, By = Ty, ASx = SAx$ and $BTy = TBy$.

Now we prove that $Ax = By$.

From (2.3.1), we have

$$M(Ax, By, ut) \geq \psi\left(min\left\{\begin{matrix} M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ M(By, Sx, t)\left(\frac{1+M(By,Ty,t)}{1+M(Ax,Sx,t)}\right), M(Ax, Ty, t)\left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right) \end{matrix}\right\} \right)$$

$$= \psi\left(min\left\{\begin{matrix} M(Ax, Ty, t), M(Ax, Ax, t), M(By, By, t), \\ M(By, Ax, t)\left(\frac{1+M(By,By,t)}{1+M(Ax,Ax,t)}\right), M(Ax, Ty, t)\left(\frac{1+M(Ax,Ax,t)}{1+M(By,By,t)}\right) \end{matrix}\right\} \right)$$

$$= \psi\left(min\left\{ M(Ax, By, t), 1,1, M(Ax, By, t)\left(\frac{1+1}{1+1}\right), M(Ax, By, t)\left(\frac{1+1}{1+1}\right) \right\} \right)$$

$$= \psi\big(M(Ax, By, t)\big)$$

Therefore $M(Ax, By, ut) \geq \psi\big(M(Ax, By, t)\big)$ for $t > 0$.

Therefore $M(Ax, By, t) \geq \psi\left(M\left(Ax, By, \frac{t}{u}\right)\right) \geq \psi^2\left(M\left(Ax, By, \frac{t}{u^2}\right)\right) \geq \cdots \geq \psi^n\left(M\left(Ax, By, \frac{t}{u^n}\right)\right)$

Since $\psi \in \Psi$, $\psi^n\left(M\left(Ax, By, \frac{t}{u^n}\right)\right) \to 1$ as $n \to \infty$.

Therefore $M(Ax, By, t) \geq 1$

Therefore $Ax = By$.

Thus $Ax = Sx = By = Ty$.

Suppose that there exists another point $z \in X \ni Az = Sz$.

Then from (2.3.1), we have $Az = Sz = By = Ty$.

Therefore $Ax = Az$ and $Sx = Sz$

Therefore $w = Ax = Sx$ is the unique point of coincidence of *A* and *S*.

Therefore by Lemma 1.9, w is the only common fixed point of *A* and *S*.

Since $Ax = Sx = By = Ty$, w is the only common fixed point of B and T.

Thus A, B, S and T have a unique common fixed point in *X*.

Note: In this Theorem, completeness of the fuzzy metric space is not used.

Now we show that Theorem 2.2 follows as a corollary from Theorem 2.3

Proof of Theorem 2.2:
If (2.2.1) is satisfied, then for any $\psi \in \Psi, u \in (0,1), x, y \in X$ and $t > 0$

$$M(Ax, By, ut) \geq min\left\{\begin{matrix} M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ M(By, Sx, t)\left(\frac{1+M(By,Ty,t)}{1+M(Ax,Sx,t)}\right), M(Ax, Ty, t)\left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right)\end{matrix}\right\}$$

$$> \psi\left(min\left\{\begin{matrix} M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ M(By, Sx, t)\left(\frac{1+M(By,Ty,t)}{1+M(Ax,Sx,t)}\right), M(Ax, Ty, t)\left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right)\end{matrix}\right\}\right)$$

So that (2.3.1) is satisfied.

Consequently from Theorem 2.3, A, B, S and T have a unique common fixed point in *X*.

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