



COMMON FIXED POINTS OF SELF MAPS SATISFYING AN INTEGRAL TYPE
IN INTUITIONISTIC FUZZY METRIC SPACE

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ABSTRACT

In this paper, we prove common fixed point theorems for weakly compatible self maps of integral type in intuitionistic fuzzy metric space. these results are proved with exploiting the notion of continuity and without imposing any condition of t - norm.

Key Words: intuitionistic fuzzy metric space, weakly compatible maps of common fixed point.

1. INTRODUCTION

Atanassove[1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park [2] defined the notion of intuitionistic fuzzy metric space with the help of continuous t -norms and continuous. Alaca *et al.*[3] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t -norm and continuous as a generalization of fuzzy metric space due to Kramosil and Michalek[4]. In 2006, Turkoglu[5] proved Jungck's[6] common fixed point theorem in the setting of intuitionistic fuzzy metric spaces for commuting mappings. Afterwards, many authors proved common fixed point theorems using different variants in such spaces. We prove common fixed point theorem for two weakly compatible maps satisfying an integral type contractive condition in intuitionistic fuzzy metric space. These results are proved without exploiting the notion of continuity and without imposing any condition of t -norm.

II. PRELIMINARIES

Definition 2.1[7]: A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ and \diamond : $[0,1] \times [0,1] \rightarrow [0,1]$ is continuous t -norm if $*$ and \diamond satisfies the following conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in [0, 1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.
5. \diamond is commutative and associative;
6. \diamond is continuous;
7. $a \diamond 0 = a$ for all $a \in [0, 1]$;
8. $a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2[3]: A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t -norm, \diamond is a continuous t -conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

- (i) $M(x, y, t) + N(x, y, t) \leq 1$ for all x, y in X and $t > 0$;
- (ii) $M(x, y, 0) = 0$ for all x, y in X ;
- (iii) $M(x, y, t) = 1$ for all x, y in X and $t > 0$ if and only if $x = y$;
- (iv) $M(x, y, t) = M(y, x, t)$ for all x, y in X and $t > 0$;
- (v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all x, y, z in X and $s, t > 0$;
- (vi) for all x, y in $X, M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous;
- (vii) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all x, y in X and $t > 0$;
- (viii) $N(x, y, 0) = 1$ for all x, y in X ;
- (ix) $N(x, y, t) = 0$ for all x, y in X and $t > 0$ if and only if $x = y$;

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- (x) $N(x, y, t) = N(y, x, t)$ for all x, y in X and $t > 0$;
- (xi) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all x, y, z in X and $s, t > 0$;
- (xii) for all x, y in X , $N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is right continuous;
- (xiii) $\lim_{t \rightarrow \infty} N(x, y, t) = 1$ for all x, y in X ;

Then (M, N) is called an intuitionistic fuzzy metric on X . The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t respectively. an intuitionistic fuzzy metric space of the form $(X, M, I-M, *, \diamond)$ such that t -norm $*$ and t -conorm \diamond are associated as $x \diamond y = I-((I-x) * (I-y))$ for all x, y in X .

Lemma 2.3: [3] In intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x, y, *)$ is non-decreasing and $N(x, y, \diamond)$ is non-increasing for all x, y in X .

Definition 2.4[3]: Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ in X is

- (i) said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n \rightarrow \infty} N(x_n, x_{n+p}, t) = 0$.
- (ii) said to be convergent to a point x in X if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$ and $\lim_{n \rightarrow \infty} N(x_n, x, t) = 0$.
- (iii) said to be complete if and only if every Cauchy sequence in X is convergent.

Example 2.5:[3] $X = \{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ and let $*$ be the continuous t-norm and commuting if \diamond be the continuous t-norm defined by $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ Respectively, for all $a, b \in [0, 1]$. for each $t \in (0, \infty)$ and $x, y \in X$, defined (M, N) by

$$M(x, y, t) \begin{cases} \frac{t}{t+|x-y|}, & t > 0 \\ 0, & t = 0 \end{cases} \text{ and } N(x, y, t) \begin{cases} \frac{t}{t+|x-y|}, & t > 0 \\ 0, & t = 0 \end{cases}$$

Clearly, $(X, M, N, *, \diamond)$ is complete intuitionistic fuzzy metric space.

Definition 2.6[5]: A pair of self mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be $M(fgx, gfx, t) = 1$ and $N(fgx, gfx, t) = 0$ for all $x \in X$.

Definition 2.7[5]: A pair of self mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be weakly commuting if

$$M(fgx, gfx, t) \geq M(fx, gx, t) \text{ and } N(fgx, gfx, t) \leq N(fx, gx, t) \text{ for all } x \in X \text{ and } t > 0.$$

Definition 2.8[5]: A pair of self mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be compatible if

$$\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t) \text{ and } \lim_{n \rightarrow \infty} N(fgx_n, gfx_n, t)$$

Definition 2.9 [8]: Two self maps f and g on a set X are said to be weakly compatible if they commute at the coincidence points i.e., if $fu = gu$ for some u in X , then $fgu = gfu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.13: A pair of self mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be weakly compatible of type (A) if

$$\lim_{n \rightarrow \infty} M(fgx_n, gfgx_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(fgx_n, gfgx_n, t) = 0$$

$$\lim_{n \rightarrow \infty} M(gfx_n, fgfx, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(gfx_n, fgfx, t) = 0 \text{ for } t > 0 \text{ whenever } \{x_n\} \text{ is sequence in } X \text{ such that}$$

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u \text{ for some } u \text{ in } X$$

III. MAIN RESULT

Theorem 3.1: let f and g be weakly compatible maps of type(A) of a complete intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ satisfying the following conditions

1. $f(x) \subseteq g(x)$
2. any one of the mappings f or g is continuous
3. $\int_0^{M(fx, fy, t)} \varphi(t) dt < c \int_0^{M(gx, gy, t)} \varphi(t) dt$
 $\int_0^{N(fx, fy, t)} \varphi(t) dt > c' \int_0^{N(gx, gy, t)} \varphi(t) dt$

For each x, y in $X, t > 0$ c belong to $[0,1]$ where $\varphi: R^+ \rightarrow R$ is a lebesgue integrable mapping which is summable non negative and such

$$4. \int_0^\epsilon \varphi(t) dt > 0 \text{ for each } \epsilon > 0$$

Then f and g have a unique common fixed point in X .

Proof: Let x_0 be an arbitrary point in $X, f(x) \sqsubseteq g(x)$ Choose a point x_1 in X such that $fx_0 = gx_1$. In general, we can choose x_{n+1} such that $fx_n = gx_{n+1}$ for all $n \geq 1$. Then for all $t > 0$

$$\begin{aligned} \int_0^{M(fx_n, fx_{n+1}, t)} \varphi(t) dt &\geq c \int_0^{M(fx_n, fx_{n+1}, t)} \varphi(t) dt \\ &= c \int_0^{M(fx_n, -1, fx_n, t)} \varphi(t) dt \\ &> c \int_0^{M(fx_n, -1, fx_n, t)} \varphi(t) dt \end{aligned} \tag{1}$$

$$\begin{aligned} \int_0^{N(fx_n, fx_{n+1}, t)} \varphi(t) dt &\leq c' \int_0^{N(fx_n, fx_{n+1}, t)} \varphi(t) dt \\ &= c' \int_0^{N(fx_n, -1, fx_n, t)} \varphi(t) dt \\ &< c' \int_0^{N(fx_n, -1, fx_n, t)} \varphi(t) dt \end{aligned} \tag{2}$$

Since $c(t) > t$ and $c'(t) < t$ for all $0 < t < 1$. $\int_0^{M(fx_n, fx_{n+1}, t)} \varphi(t) dt$ is an increasing sequence of positive real number in $[0,1]$ and $\int_0^{N(fx_n, fx_{n+1}, t)} \varphi(t) dt$ is decreasing sequence of positive real number in $[0,1]$. therefore they converges to limits $L \leq 1$, and $L' \geq 0$ respectively.

Now, we claim that $L=1$ and $L' = 0$. For, let $L < 1$. letting $n \rightarrow \infty$ in (1), we have $L \geq c(L) > L$, which is a contradiction and so $L=1$. Similarly let $L' \geq 0$

letting $n \rightarrow \infty$ in (2), we have $L' \leq c(L') \leq L'$ which is a contradiction and so $L' = 0$

Now, for any positive integer p and $t > 0$ we have

$$\begin{aligned} \int_0^{M(fx_n, fx_{n+p}, t)} \varphi(t) dt &\geq \int_0^{M(fx_n, fx_{n+1}, \frac{t}{p}) * \dots * M(fx_{n+p-1}, fx_{n+p}, \frac{t}{p})} \varphi(t) dt \\ &\geq \int_0^{M(fx_n, fx_{n+1}, \frac{t}{p}) * \dots * M(fx_n, fx_{n+1}, \frac{t}{p})} \varphi(t) dt \end{aligned}$$

$$\begin{aligned} \int_0^{N(fx_n, fx_{n+p}, t)} \varphi(t) dt &\leq \int_0^{N(fx_n, fx_{n+1}, \frac{t}{p}) \diamond \dots \diamond N(fx_{n+p-1}, fx_{n+p}, \frac{t}{p})} \varphi(t) dt \\ &\leq \int_0^{N(fx_n, fx_{n+1}, \frac{t}{p}) \diamond \dots \diamond N(fx_n, fx_{n+1}, \frac{t}{p})} \varphi(t) dt \end{aligned}$$

Since we have

$$\lim_{n \rightarrow \infty} \int_0^{M(fx_n, fx_{n+p}, \frac{t}{p})} \varphi(t) dt = 1$$

$$\lim_{n \rightarrow \infty} \int_0^{N(fx_n, fx_{n+p}, \frac{t}{p})} \varphi(t) dt = 0$$

It follows that

$$\lim_{n \rightarrow \infty} \int_0^{M(fx_n, fx_{n+p}, t)} \varphi(t) dt \geq 1 * \dots * 1 \geq 1$$

$$\lim_{n \rightarrow \infty} \int_0^{N(fx_n, fx_{n+p}, t)} \varphi(t) dt \leq 0 \diamond \dots \diamond 0 \leq 0$$

Thus by definition 2.5 $\{fx_n\}$ is a Cauchy sequence and by the completeness of X . $\{fx_n\}$ converges to a point z belong to X . Also $\{gx_n\}$ converges to the point z .

Suppose that, by the mapping f is continuous. Then $\lim_{n \rightarrow \infty} ffx_n = fz$, Further, since f and g are weakly commuting,

We have

$$\int_0^{M(fg, gfx_n, t)} \varphi(t) dt \geq \int_0^{M(fx_n, gx_n, \frac{t}{R})} \varphi(t) dt$$

$$\int_0^{N(fg, gfx_n, t)} \varphi(t) dt \leq \int_0^{N(fx_n, gx_n, \frac{t}{R})} \varphi(t) dt$$

Letting $n \rightarrow \infty$ in the inequality we have $\lim_{n \rightarrow \infty} gfx_n = fz$.

Now, we prove that $z=fz$. Suppose $z \neq fz$. Then there exist $t > 0$ such that

$$\int_0^{M(z, fz, t)} \varphi(t) dt \leq 1 \text{ and, } \int_0^{N(z, fz, t)} \varphi(t) dt \geq 0$$

We have

$$\int_0^{M(fx_n, ffx_n, t)} \varphi(t) dt \geq C \int_0^{M(gx_n, gfx_n, t)} \varphi(t) dt$$

$$\int_0^{N(fx_n, ffx_n, t)} \varphi(t) dt \geq C' \int_0^{N(gx_n, gfx_n, t)} \varphi(t) dt$$

Letting $n \rightarrow \infty$ in the above inequalities, we have

$$\int_0^{M(z, fz, t)} \varphi(t) dt \geq c \int_0^{M(z, fz, t)} \varphi(t) dt \geq \int_0^{M(z, fz, t)} \varphi(t) dt$$

$$\int_0^{N(z, fz, t)} \varphi(t) dt \leq c' \int_0^{N(z, fz, t)} \varphi(t) dt \leq \int_0^{N(z, fz, t)} \varphi(t) dt$$

Which are a contradiction, therefore $z=fz$, since we can find a point $z_1 \in X$ such $z=fz=gz_1$. now, it follow that

$$\int_0^{M(ffx_n, fz_1, t)} \varphi(t) dt \geq C \int_0^{M(gfx_n, gz_1, t)} \varphi(t) dt$$

$$\int_0^{N(ffx_n, fz_1, t)} \varphi(t) dt \geq c' \int_0^{N(gfx_n, gz_1, t)} \varphi(t) dt$$

Letting $n \rightarrow \infty$ in the above inequalities, we have

$$\int_0^{M(fz, fz_1, t)} \varphi(t) dt \geq c \int_0^{M(fz, gz_1, t)} \varphi(t) dt = 1$$

$$\int_0^{N(fz, fz_1, t)} \varphi(t) dt \leq c' \int_0^{N(fz, gz_1, t)} \varphi(t) dt = 0$$

Which implies that $fz_1=gz_1$. also, for any $t > 0$

$$\int_0^{M(fz, gz, t)} \varphi(t) dt = \int_0^{M(fgz_1, gfgz_1, t)} \varphi(t) dt \geq \int_0^{M(fz_1, gz_1, \frac{t}{R})} \varphi(t) dt = 1$$

$$\int_0^{N(fz, gz, t)} \varphi(t) dt = \int_0^{N(fgz_1, gfgz_1, t)} \varphi(t) dt \leq \int_0^{N(fz_1, gz_1, \frac{t}{R})} \varphi(t) dt = 0$$

Which again imply that $fz=gz$, therefore z is a common fixed point of f and g . Which to prove the uniqueness of the common fixed point z , let y ($y \neq z$) be another common fixed point of f and g . Then there exists $t > 0$ such that

$$\int_0^{M(z, y, t)} \varphi(t) dt \leq 1. \text{ and } \int_0^{N(z, y, t)} \varphi(t) dt \geq 0.$$

$$\int_0^{M(z, y, t)} \varphi(t) dt = \int_0^{M(fz, fy, t)} \varphi(t) dt \geq c \int_0^{M(gz, gy, t)} \varphi(t) dt = c \int_0^{M(z, y, t)} \varphi(t) dt \geq \int_0^{M(z, y, t)} \varphi(t) dt$$

$$\int_0^{N(z, y, t)} \varphi(t) dt = \int_0^{N(fz, fy, t)} \varphi(t) dt \geq c' \int_0^{N(gz, gy, t)} \varphi(t) dt = c' \int_0^{N(z, y, t)} \varphi(t) dt \leq \int_0^{N(z, y, t)} \varphi(t) dt$$

Which is a contradiction since $c(t) > t$ and $c'(t) < t$ for any $0 < t < 1$. Therefore $z=y$, i.e z is a unique common fixed point of f and g . this completes the proof.

Example3.1:[3] $X = \{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ and let $*$ be the continuous t-norm and commuting if \diamond be the continuous t-norm defined by $a*b = ab$ and $a \diamond b = \min\{1, a+b\}$ respectively, for all $a, b \in [0, 1]$. for each $t \in (0, \infty)$ and $x, y \in X$, defined (M, N) by

$$M(x, y, t) \begin{cases} \frac{t}{t+|x-y|}, & t > 0 \\ 0, & t = 0 \end{cases} \text{ and } N(x, y, t) \begin{cases} \frac{t}{t+|x-y|}, & t > 0 \\ 0, & t = 0 \end{cases}$$

Clearly, $(X, M, N, *, \diamond)$ is complete intuitionistic fuzzy metric space. where $*$ is defined by $a*b = ab$ and $a \diamond b = \min\{1, a+b\}$. Further, $(X, M, N, *, \diamond)$ is complete.

Define

$$F(x) = 1, g(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is a irrational number} \end{cases}$$

It is evident that $f(x) \equiv g(x)$, f is continuous and g is discontinuous. Define a function $C: [0, 1] \rightarrow [0, 1]$ by $c(t) = \sqrt{t}$ for any $0 < t < 1$ and $C(t) = 1$ for $t = 1$.

$c': [0, 1] \rightarrow [0, 1]$ by $c'(t) = t^2$ for any $0 < t < 1$ and $C'(t) = 0$ for $t = 0$. Then $c(t) > t$, $c'(t) < t$ for any $0 < t < 1$ and $M(fx, fy, t) \geq cN(gx, gy, t)$ and $N(fx, fy, t) \geq c'N(gx, gy, t)$

For all x, y in X . further, f and g are R -weakly commuting. Thus all the conditions of theorem are satisfied and is a common fixed of f and g .

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