International Research Journal of Pure Algebra -3(6), 2013, 237-241
Available online through www.rjpa.info ISSN 2248–9037

GENERALIZED ON A COMMON FIXED POINTS IN FUZZY METRIC SPACES

M. Vijaya Kumar[1], P. Devidas[2*] and Savitha Jadhav[3]

[1,3]*Department of Mathematics, Vaagdevi College of Engineering, Warangal, (A.P.), India*

[2]*Bhagwant University, India*

(Received on: 30-05-13; Revised & Accepted on: 11-06-13)

ABSTRACT

In this paper, we give generalized on common fixed points in fuzzy metric space. our results extended and generalized fixed point theorem on complete fuzzy metric spaces.

Mathematics Subject Classification: *54H25, 54E20.*

Keywords: *compatible mappings, common fixed point, fuzzy metric space.*

1. INTRODUCTION

Introduced the concept of fuzzy metric spaces in different ways. In [3, 4], George and Veeramani modified the concept of fuzzy metric space which introduced by Kramosil and Michalek [10]. They, also, obtained the Hausdorff topology for this kind of fuzzy metric spaces and showed that every metric induces a fuzzy metric. Sessa [12] introduced a generalization of commutativity, so called weak commutativity. Further Jungck [7] introduced more generalized commutativity, which is called compatibility in metric space. He proved common fixed point theorems. Recently, Bijendra Singh and M. S. Chauhan [13] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric spaces in the sence of George and Veeramani with continuous t-norm * defined by $a * b = min\{a, b\}$ for all a, b * [0, 1].In this paper we modify common fixed point theorems obtained in [13] and we characterize the conditions for two continuous self mappings of complete fuzzy metric space have a unique common fixed point.

2. PRELIMINARIES

Definition 2.1: *[11] A binary operation* $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ *is called a continuous t-norm if* ([0, 1], *) *is an abelian topological monoid with* 1 *such that* $a * b \leq c * d$, *whenever* $a \leq c$, $b \leq d$ *for all* $a, b, c, d \in [0, 1]$. Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2: *[3] The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if X is an arbitrary set, *is a continuous t-norm and M is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions:*

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, *\cdot): (0, \infty) \rightarrow [0, 1]$ *is continuous, for all* $x, y, z \in X$ *and* $t, s > 0$.

Let (X, d) be a metric space, and let $a * b = ab$ or $a * b = \min\{a, b\}$. Let $M(x, y, t) = t/t + d(x, y)$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [3].

Definition 2.3: *[5] A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is said to be convergent to a point* $x \in X$ *(denoted by* $\lim_{n\to\infty} x_n = x$*), if for each* $\epsilon > o$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - \epsilon$ *for all* $n \geq n_0$..a sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to a point $x \in X$ if and only if $\lim_{n\to\infty} M(xn, x, t) = 1$.

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M*)$ is called Cauchy sequence if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_{n+p}, t) > 1 - \epsilon$ for all $n \geq n_0$ and all $t > 0$.A fuzzy metric space in which every Cauchy sequene is convergent is saidto be complete.

Corresponding author: P. Devidas[2]
[2]Bhagwant University, India

George and Veeramani [3] give an example that (R, M, *) is not complete in the sense of [5], where M is the standard fuzzy metric with $d(x, y) = |x - y|$, and so to make R complete fuzzy metric space George and Veeramani redefine cauchy sequen

Definition 2.4: *[3] A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is called Cauchy sequence if for each* $\forall\varepsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in$ N *such that* $M(x_n, x_m, t) > 1 - \varepsilon$ *for all* $n, m, \geq n_0$.

Definition 2.5: *[13] Self mappings ε A and B of a fuzzy metric space* $(X, M, *)$ *is said to be compatible if* $\lim_{n\to\infty}$ $M(ABxn, BAxn, t) = 1$ *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ *for some* $z\ \varepsilon\ X$.

From now on, let $(X, M, *)$ be a fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y\ \varepsilon\ X$ and $s * s \geq s$ for all s [ε 0, 1]. *for all* $n, m \geq n_0$.

Lemma 2.6: *[5] Let* $(X, M, *)$ *be a fuzzy metric space. Then for*
1. *all* $x, y \in X$, $M(x, y, *)$ *is non decreasing.*
2. *If there exists* $q \in (0, 1)$ *such that* $M(x, y, qt) \geq M(x, y, t)$ *for all* $x, y \in X$ *and* $t > 0$, *then* $x = y$.
3. *let A and S be continuous self mappings of X and* [A, S] *be compatible. Let* $\{x_n\}$ *be a sequence in X such that* $Ax_n \to z$ *and* $Sx_n \to z$. *Then* $ASx_n \to Sz$.

Lemma 2.7: *[9] The only t-norm * satisfying* $r * r \geq r$ *for all* $r \in [0, 1]$ *is the minimum t-norm, that is,* $a * b = \min\{a, b\}$ *for all* $a, b \in [0, 1]$.

3. COMMON FIXED POINT THEOREMS

Let (X, M*) be complete fuzzy metric space and let A, B, S and T be self mappings on X such that the following conditions are satisfied
1. AX⊑ TX, BS⊑ SX
2. S and T are continuous
3. The pair [A, S] and [B,T] are compactable.
4. there exist k∈(0,1) such that for ever x,y∈ X and t>0

F(M(Sx,Ty,Kt)*M(Ax,By,t)*(M(Sx,Ax,t)*M(Ty,Ny,t)*M(Sx,By,t)*M(Ty,Ax,t) ≥ 1.

Then A, B, S and T have a unique common fixed point in X.

Proof: Let x_0 be arbitrary point of X. From (1) we can construct a sequence $\{y_n\}$ In X as follow:

y_{2n+1}=Sx_{2n} =Bx_{2n+1} and y_{2n+2}= STx_{2n+1} =Ax_{2n+2} for all n=0, 1, 2…

Then by (4), we have, for any t > 0

F(M(Sx_{2n}, Tx_{2n+1}, kt)* M(Ax_{2n}, Bx_{2n+1}, t)* M(Sx_{2n}, Ax_{2n}, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(Tx_{2n}, Bx_{2n+1}, t)* M(Tx_{2n+1}, Ax_{2n}, t)) ≥ 1

And so

F(M(Sx_{2n}, Tx_{2n+1}, kt)* M(Ax_{2n-1}, Bx_{2n}, t)* M(Sx_{2n}, Ax_{2n-1}, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(Tx_{2n+1}, S$x_{2n}, \frac{t}{2}$)* M(,Sx_{2n} T$x_{2n-1}, \frac{t}{2}$)) ≥ 1

By (F-2).we have
M(Sx_{2n}, Tx_{2n+1}, ht) ≥M(Sx_{2n}, Tx_{2n-1}, t)*M (,Sx_{2n} Tx_{2n+1}, t))

And so
M (,y_{2n+1} y_{2n+2}, ht) ≥M(, y_{2n+1} y_{2n}, t)*M(y_{2n+1} y_{2n+2}, t))

Which implies that
M (,y_{2n+1} y_{2n+2}, ht) ≥M(, y_{2n+1} y_{2n}, t)=M(y_{2n} y_{2n+1}, t))

Again by (F-2) we have
M(, y_{2n+1} y_{2n}, ht) ≥ M(y_{2n} y_{2n-1}, t))

In general, we have for all m=1, 2… and t >0

1. M $(y_{m+1}\ y_{m+2}, ht)$ ≥M $(y_{m+1}\ y_m, t)$=M$(y_m\ y_{m+1}, t)$)

To prove that $\{y_n\}$ is a cauchy sequence, first we prove that for any 0<$\lambda < 1$ and t>0

2. M $(y_{n+1}\ y_{n+m+1}, t)$>1-λ

For all n≥ n_0 and m∈N. here we use induction from (1), we have
M $(y_{n+1}\ y_{n+2}, t)$ ≥M $(y_n\ y_{n+1}, \frac{t}{h})$ ≥ ⋯ … … ≥M $(y_1\ y_2, \frac{t}{h^n})$) ≥1-$\lambda$

Hence (2) is true for m+1 ∈N. Thus $\{y_n\}$ is Cauchy sequence in X. Since (X, M,*) is complete, $\{y_n\}$ converges to a point z∈X. Since $\{ Ax_{2n+2}\}$,$\{ Bx_{2n+1}\}$,$\{ Sx_{2n}\}$,$\{ Tx_{2n+1}\}$→z as n→ ∞

Now, suppose that a is continuous, then the sequence $\{ Asx_{2n}\}$ converges to AZ as n→ ∞ notice that for any t>0

F(M(ASx_{2n}, Tx_{2n+1}, kt)* M(AAx_{2n}, Bx_{2n+1}, t)* M(SAx_{2n},A Ax_{2n}, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(SAx_{2n}, Bx_{2n+1}, t)* M(Tx_{2n+1}, AAx_{2n}, t)) ≥ 1

And then, by letting n→ ∞, since F is continuous, we have

F(M(Az, z, Kt)*M(Az, z, t)*1,1*M(Az, z, t)* M(Az, z, t) ≥1.

Therefore from, (f-3), we have M(Az, z, Kt) ≥ M(Az, z, t)

We have AZ=z. further more by (iv) we have

F(M(Sz, Tx_{2n+1}, kt)* M(Az, Bx_{2n+1}, t)* M(Az, sz, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(Sz, Bx_{2n+1}, t)* M(Tx_{2n+1}, Az, t)) ≥ 1

And, letting n→ ∞
F(M(Sz, z, Kt)*1,*M(Sz, z, t)*1* M(Sz, z, t)*1) ≥1.

On the other hand since
M(Sz, z, t)* ≥ M(Sz, z, $\frac{t}{2}$) = M(Sz, z, $\frac{t}{2}$)*1

And F is none increasing in the fifth variable, we have, for any t > 0

F (M (Sz, z, Kt)*1, *M(Sz, z, t)*1* M(Sz, z, $\frac{t}{2}$)*1) ≥

F(M(Sz, z, Kt)*1,*M(Sz, z, t)*1* M(Sz, z, t)*1) ≥ 1

Which implies by (F-2) that Sz=z. this means that z is the range of S and since S(x)⊑B(X), there exists a point u∈X such that Bu=z. Using (iv) we have successively

F(M(Sz, Tu, Kt)*M(Az, Bu, t)* M(Sz, Az, t)*, M(Tu, Bu, t)* M(Sz, Bu, t)* M(Tz, Az, t) ∗≥1.

F(M(z, Tu, Kt)*1*1* M(z, Tu, t)*,1*M(z, tu, t)* ∗≥1.

Which implies by (F-2) that z=Tu, since Bu=Tu=z and B, T are compatible of type (α). we have TTu=Btu soTz=TTu=Btu=Bz, therefore, from (iv) we have for any t>0,

F(M(Sz, Tz, Kt)*M(Az, Bz, t)* M(Sz, Az, t)*,M(Tu, Bz, t)* M(Sz, Bu, t)* M(Tz, Az, t) ∗≥1.

F(M(z, Tz, Kt)*M(z, Bz, t)* 1*,1* M(z, Tz, t)* M(z, Tz, t) ∗≥1.

Thus from (T-3), we have M(z, Tz, Kt) ≥M(z, Tz, t), again from we have z=Tz=Bz. consequently, z is a common fixed point of S, T, A and B. The same result holds, if we assume that B is continous instead of A.

Now, suppose that S is continuous, then the sequence {SAx_{2n}} converges to Sz as n→ ∞, notice that, for any t>0.
F(M(ASx_{2n},Sz,t) ≥M(ASx_{2n},SSx_{2n},$\frac{t}{2}$) *1* M(SS$x_{2n}$$Sz$,$\frac{t}{2}$)

Now, since S is continuous and S, A are compatible of type (α) letting n→ ∞,, we deduce that the sequence {ASx_{2n}} converges to Sz, using (iv) we have for any t>0

F(M(SSx_{2n}, Tx_{2n+1}, kt)* M(ASx_{2n}, Bx_{2n+1}, t)* M(SSx_{2n},A AS, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(SSx_{2n}, Bx_{2n+1}, t)* M(Tx_{2n+1}, ASx_{2n}, t)) ≥ 1

And then, by letting n→ ∞, since F is continuous, we have

F(M(Sz, z, Kt)*M(Sz, z, t)* 1*,1* M(Sz, z, t)* M(Sz, z, t) ∗≥1.

Thus from (F-3) we have M (Sz, z, Kt) ≥M(Sz, Tz, t) again from we have Sz=z. This means that z is the range of S and since S(X)⊑B(X) there exists a point vbelong X such that nBv=z . Using (iv), we have for any t>0

F (M(SSx_{2n}, Tv, kt)* M(ASx_{2n}, Bv, t)* M(SSx_{2n}, ASx_{2n}, t)* M(Tv, Bv, t)* M(SSx_{2n}, Bv, t)* M(Tv, ASx_{2n}, t) ≥ 1

letting n→∞,

F(M(z, Tv, Kt)* 1*,1* M(z, Tv, t)*1* M(z, Tv, t) ∗≥1.

Which implies by (F-2) that z=Tv, since Bv==Tv=z and B, T are compatible of type (α) we have TBv=BBv and so Tz=TBv=BBv=Bz. Thus from (iv), we have

F(M(Sx_{2n}, Tz, kt)* M(Ax_{2n}, Bz, t)* M(Sx_{2n}, Ax_{2n}, t)* M(Tz, Bz, t)* M(Sx_{2n}, Bz, t)* M(Tv, Ax_{2n}, t)) ≥ 1

Letting n→∞,

F(M(z, Tz, Kt) M(z, Tz, t)* 1*, 1* M(z, Tz, t)* M(z, Tz, t) ∗≥1.

Thus z=Tz=Bz. This means that z is the range of T and since T(x) ⊑ $A(X)$, there exists w belong sto X such that Aw=z.

Thus from (iv) we have for any t>0.

F(M(Sw, T, z, Kt)*M(Aw, Bz_1, t)* M(Sw, Aw, t)* M(Tz, Bz, t)* M(Sw, Bz, t) ∗ M(Tz, Aw, t) ≥1.

F(M(Sw, z, Kt)*1* M(Sw, z, t)* 1* M(Sw, z, t) ∗ 1) ≥1

And by (F-2) we have z=Sw =Aw =z and S, A, are compatible of type (α) we have z=Sz+Saw=AAw=Az and thus z=Az. consequently, z is a common fixed point S, T, A and B. The same results holds if we assume that T is continuous instead of S.

Finally we show that the point z is unique common fixed point of S, T, A, B. Suppose that S, T, A and B have another common fixed point $z_1$theny, by (iv) we have, for any t>0.

F(M(Sz, Tz_1, kt)* M(Az, Bz_1, t)* M(Sz, Az, t)* M(Tz_1, B$z_1 t$)* M(Sz, Bz_1, t)* M(Tz_1, Az, t)) ≥ 1

F(M(z,, z_1,Kt)* (M(z,, z_1,t) 1*,1* M(z, z_1,t)*1* M(z, z_1, t) ∗≥1

Thus from (F-3) we have (M (z,, z_1K, t) ≥ (M(z,, z_1,t)we have z=z_1. This completes the proof.

Corollary 3.2: Let (X, M*) be complete fuzzy metric space and let A, B, S and T be self mappings on X such that the following conditions are satisfied (i)-(iii) and theorem 3.1. there exist k∈(0,1) such that for ever x, y∈ X and t>0

F(M(SX,TY,Kt)*M(AX,BY,t)*(M(SX,AX,t)*M(TY,NY,t)*M(SX,BY,t)*M(TY,AX,t) ≥ 1.

Then A, B, S and T have a unique common fixed point in X.

REFERENCES

[1] Deng Zi-ke, Fuzzy pseudo metric spaces, *J. Math. Anal. Appl.*, 86 (1982), 74-95.

[2] M. A. Erceg, Metric spaces in fuzzy set theory, *J. Math. Anal. Appl.*, 69(1979), 205-230.

[3] A. George and P. Veeramani, On some results in fuzzy metirc spaces, *Fuzzy sets and Systems*, 64(1994), 395-399.

[4] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy sets and Systems*, 90(1997), 365-368.

[5] M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy sets and Systems*, 27(1988), 385-389.

[6] Valentin Gregori and Almanzor Sapena, On fixed point theorems in fuzzy metric spaces, *Fuzzy sets and Systems*, 125(2002), 245-252.

[7] G. Jungck, Compatible mappings and fixed points, *Internat. J. Math. Math. Sci.*, 9(4) (1986), 771-779.

[8] O. Kaleva and S. Seikkala, On fuzzy metric spaces, *Fuzzy sets and Systems*, 12(1984), 215-229.

[9] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer Academic Publishers.

[10] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, 11(1975), 326-334.

[11] B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.*, 10(1960), 314-334.

[12] S. Sessa, On weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. Beograd*, 32(46)(1982), 149-153.

[13] Bijendra Singh and M. S. Chauhan, Common fixed points of compatible maps in fuzzy metric spases, *Fuzzy sets and Systems*, 115(2000), 471-475.

[14] L. A. Zadeh, Fuzzy sets, *Inform. and Control*, 8(1965), 338-353.

[15] D.O. Hebb, *Organization of Behaviour*, Wiley, New York, 1949.

Source of support: Nil, Conflict of interest: None Declared