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GENERALIZED ON A COMMON FIXED POINTS IN FUZZY METRIC SPACES

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ABSTRACT

In this paper, we give generalized on common fixed points in fuzzy metric space. our results extended and generalized fixed point theorem on complete fuzzy metric spaces.

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Keywords: *compatible mappings, common fixed point, fuzzy metric space.*

1. INTRODUCTION

Introduced the concept of fuzzy metric spaces in different ways. In [3, 4], George and Veeramani modified the concept of fuzzy metric space which introduced by Kramosil and Michalek [10]. They, also, obtained the Hausdorff topology for this kind of fuzzy metric spaces and showed that every metric induces a fuzzy metric. Sessa [12] introduced a generalization of commutativity, so called weak commutativity. Further Jungck [7] introduced more generalized commutativity, which is called compatibility in metric space. He proved common fixed point theorems. Recently, Bijendra Singh and M. S. Chauhan [13] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric spaces in the sence of George and Veeramani with continuous t-norm * defined by $a * b = min\{a, b\}$ for all a, b * [0, 1].In this paper we modify common fixed point theorems obtained in [13] and we characterize the conditions for two continuous self mappings of complete fuzzy metric space have a unique common fixed point.

2. PRELIMINARIES

Definition 2.1: *[11] A binary operation* $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ *is called a continuous t-norm if* $([0, 1], *)$ *is an abelian topological monoid with* 1 *such that* $a * b \leq c * d$, *whenever* $a \leq c$, $b \leq d$ *for all* $a, b, c, d \in [0, 1]$. Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2: *[3] The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if X is an arbitrary set, *is a continuous t-norm and M is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions:*

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, *\cdot): (0, \infty) \rightarrow [0, 1]$ *is continuous, for all* $x, y, z \in X$ *and* $t, s > 0$.

Let (X, d) be a metric space, and let $a * b = ab$ or $a * b = \min\{a, b\}$. Let $M(x, y, t) = t/t + d(x, y)$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [3].

Definition 2.3: *[5] A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is said to be convergent to a point* $x \in X$ *(denoted by* $\lim_{n\rightarrow\infty} x_n = x$*), if for each* $\epsilon > o$ *and each* $t > 0$, *there exists* $n_0 \in$ N *such that* $M(x_n, x, t) > 1 - \epsilon$ *for all* $n \geq n_0$..a sequence $\{x_n\}$ in a fuzzy metric space $(X,M, *)$ converges to a point $x \in X$ if and only if $\lim_{n\rightarrow\infty} M(xn, x, t) = 1$.

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M*)$ is called Cauchy sequence if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in$ N such that $M(x_n, x_{n+p}, t) > 1 - \epsilon$ for all $n \geq n_0$ and all $t > 0$.A fuzzy metric space in which every Cauchy sequene is convergent is saidto be complete.

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George and Veeramani [3] give an example that (R, *M*, *) is not complete in the sense of [5], where *M* is the standard fuzzy metric with $d(x, y) = |x - y|$, and so to make R complete fuzzy metric space George and Veeramani redefine cauchy sequen

Definition 2.4: *[3] A sequence* $\{x_n\}$ *in a fuzzy metric space* (*X, M,* *) *is called Cauchy sequence if for each* $\forall\varepsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in$ N *such that* $M(x_n, x_m, t) > 1 - \varepsilon$ *for all* $n, m, \geq n_0$.

Definition 2.5: *[13] Self mappings ε A and B of a fuzzy metric space* (*X, M,* *) *is said to be compatible if* $\lim_{n\to\infty}$ $M(ABxn, BAxn, t) = 1$ *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ *for some* $z\ \varepsilon\ X$.

From now on, let (*X, M,* *) be a fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y\ \varepsilon\ X$ and $s * s \geq s$ for all s [ε 0, 1]. *for all* $n, m \geq n_0$.

Lemma 2.6: *[5] Let* (*X, M,* *) *be a fuzzy metric space. Then for*
1. *all* $x, y \in X$, $M(x, y, *)$ *is non decreasing.*
2. *If there exists* $q \in (0, 1)$ *such that* $M(x, y, qt) \geq M(x, y, t)$ *for all* $x, y \in X$ *and* $t > 0$, *then* $x = y$.
3. *let A and S be continuous self mappings of X and* [*A, S*] *be compatible. Let* $\{x_n\}$ *be a sequence in X such that* $Ax_n \to z$ *and* $Sx_n \to z$. *Then* $ASx_n \to Sz$.

Lemma 2.7: *[9] The only t-norm * satisfying* $r * r \geq r$ *for all* $r \in [0, 1]$ *is the minimum t-norm, that is,* $a * b = \min\{a, b\}$ *for all* $a, b \in [0, 1]$.

3. COMMON FIXED POINT THEOREMS

Let (X, M*) be complete fuzzy metric space and let A, B, S and T be self mappings on X such that the following conditions are satisfied
1. AX⊑ TX, BS⊑ SX
2. S and T are continuous
3. The pair [A, S] and [B,T] are compactable.
4. there exist k∈(0,1) such that for ever x,y∈ X and t>0

F(M(Sx,Ty,Kt)*M(Ax,By,t)*(M(Sx,Ax,t)*M(Ty,Ny,t)*M(Sx,By,t)*M(Ty,Ax,t) ≥ 1.

Then A, B, S and T have a unique common fixed point in X.

Proof: Let x_0 be arbitrary point of X. From (1) we can construct a sequence $\{y_n\}$ In X as follow:

y_{2n+1}=Sx_{2n} =Bx_{2n+1} and y_{2n+2}= STx_{2n+1} =Ax_{2n+2} for all n=0, 1, 2…

Then by (4), we have, for any t > 0

F(M(Sx_{2n}, Tx_{2n+1}, kt)* M(Ax_{2n}, Bx_{2n+1}, t)* M(Sx_{2n}, Ax_{2n}, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(Tx_{2n}, Bx_{2n+1}, t)* M(Tx_{2n+1}, Ax_{2n}, t)) ≥ 1

And so

F(M(Sx_{2n}, Tx_{2n+1}, kt)* M(Ax_{2n-1}, Bx_{2n}, t)* M(Sx_{2n}, Ax_{2n-1}, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(Tx_{2n+1}, S$x_{2n}, \frac{t}{2}$)* M(,Sx_{2n} T$x_{2n-1}, \frac{t}{2}$)) ≥ 1

By (F-2).we have
M(Sx_{2n}, Tx_{2n+1}, ht) ≥M(Sx_{2n}, Tx_{2n-1}, t)*M (,Sx_{2n} Tx_{2n+1}, t))

And so
M (,y_{2n+1} y_{2n+2}, ht) ≥M(, y_{2n+1} y_{2n}, t)*M(y_{2n+1} y_{2n+2}, t))

Which implies that
M (,y_{2n+1} y_{2n+2}, ht) ≥M(, y_{2n+1} y_{2n}, t)=M(y_{2n} y_{2n+1}, t))

Again by (F-2) we have
M(, y_{2n+1} y_{2n}, ht) ≥ M(y_{2n} y_{2n-1}, t))

In general, we have for all m=1, 2… and t >0

1. M ($y_{m+1}\ y_{m+2}, ht$) ≥M ($y_{m+1}\ y_m, t$)=M($y_m\ y_{m+1}, t$))

To prove that $\{y_n\}$ is a cauchy sequence, first we prove that for any 0<$\lambda < 1$ and t>0

2. M ($y_{n+1}\ y_{n+m+1}, t$)>1-λ

For all n≥ n_0 and m∈N. here we use induction from (1), we have
M ($y_{n+1}\ y_{n+2}, t$) ≥M ($y_n\ y_{n+1}, \frac{t}{h}$) ≥ ⋯ … … ≥M ($y_1\ y_2, \frac{t}{h^n}$)) ≥1-$\lambda$

Hence (2) is true for m+1 ∈N. Thus $\{y_n\}$ is Cauchy sequence in X. Since (X, M,*) is complete, $\{y_n\}$ converges to a point z∈X. Since { Ax_{2n+2}},{ Bx_{2n+1}},{ Sx_{2n}},{ Tx_{2n+1}}→z as n→ ∞

Now, suppose that a is continuous, then the sequence { Asx_{2n}} converges to AZ as n→ ∞ notice that for any t>0

F(M(ASx_{2n}, Tx_{2n+1}, kt)* M(AAx_{2n}, Bx_{2n+1}, t)* M(SAx_{2n},A Ax_{2n}, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(SAx_{2n}, Bx_{2n+1}, t)* M(Tx_{2n+1}, AAx_{2n}, t)) ≥ 1

And then, by letting n→ ∞, since F is continuous, we have

F(M(Az, z, Kt)*M(Az, z, t)*1,1*M(Az, z, t)* M(Az, z, t) ≥1.

Therefore from, (f-3), we have M(Az, z, Kt) ≥ M(Az, z, t)

We have AZ=z. further more by (iv) we have

F(M(Sz, Tx_{2n+1}, kt)* M(Az, Bx_{2n+1}, t)* M(Az, sz, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(Sz, Bx_{2n+1}, t)* M(Tx_{2n+1}, Az, t)) ≥ 1

And, letting n→ ∞
F(M(Sz, z, Kt)*1,*M(Sz, z, t)*1* M(Sz, z, t)*1) ≥1.

On the other hand since
M(Sz, z, t)* ≥ M(Sz, z, $\frac{t}{2}$) = M(Sz, z, $\frac{t}{2}$)*1

And F is none increasing in the fifth variable, we have, for any t > 0

F (M (Sz, z, Kt)*1, *M(Sz, z, t)*1* M(Sz, z, $\frac{t}{2}$)*1) ≥

F(M(Sz, z, Kt)*1,*M(Sz, z, t)*1* M(Sz, z, t)*1) ≥ 1

Which implies by (F-2) that Sz=z. this means that z is the range of S and since S(x)⊑B(X), there exists a point u∈X such that Bu=z. Using (iv) we have successively

F(M(Sz, Tu, Kt)*M(Az, Bu, t)* M(Sz, Az, t)*, M(Tu, Bu, t)* M(Sz, Bu, t)* M(Tz, Az, t) ∗≥1.

F(M(z, Tu, Kt)*1*1* M(z, Tu, t)*,1*M(z, tu, t)* ∗≥1.

Which implies by (F-2) that z=Tu, since Bu=Tu=z and B, T are compatible of type (α). *we* have TTu=Btu soTz=TTu=Btu=Bz, therefore, from (iv) we have for any t>0,

F(M(Sz, Tz, Kt)*M(Az, Bz, t)* M(Sz, Az, t)*,M(Tu, Bz, t)* M(Sz, Bu, t)* M(Tz, Az, t) ∗≥1.

F(M(z, Tz, Kt)*M(z, Bz, t)* 1*,1* M(z, Tz, t)* M(z, Tz, t) ∗≥1.

Thus from (T-3), we have M(z, Tz, Kt) ≥M(z, Tz, t), again from we have z=Tz=Bz. consequently, z is a common fixed point of S, T, A and B. The same result holds, if we assume that B is continous instead of A.

Now, suppose that S is continuous, then the sequence {SAx_{2n}} converges to Sz as n→ ∞, notice that, for any t>0.
F(M(ASx_{2n},Sz,t) ≥M(ASx_{2n},SSx_{2n},$\frac{t}{2}$) *1* M(SS$x_{2n}Sz$,$\frac{t}{2}$)

Now, since S is continuous and S, A are compatible of type (α) letting n→ ∞,, we deduce that the sequence {ASx_{2n}} converges to Sz, using (iv) we have for any t>0

F(M(SSx_{2n}, Tx_{2n+1}, kt)* M(ASx_{2n}, Bx_{2n+1}, t)* M(SSx_{2n},A AS, t)* M(Tx_{2n+1}, Bx_{2n+1}, t)* M(SSx_{2n}, Bx_{2n+1}, t)* M(Tx_{2n+1}, ASx_{2n}, t)) ≥ 1

And then, by letting n→ ∞, since F is continuous, we have

F(M(Sz, z, Kt)*M(Sz, z, t)* 1*,1* M(Sz, z, t)* M(Sz, z, t) *≥1.

Thus from (F-3) we have M (Sz, z, Kt) ≥M(Sz, Tz, t) again from we have Sz=z. This means that z is the range of S and since S(X)⊑B(X) there exists a point vbelong X such that nBv=z . Using (iv), we have for any t>0

F (M(SSx_{2n}, Tv, kt)* M(ASx_{2n}, Bv, t)* M(SSx_{2n}, ASx_{2n}, t)* M(Tv, Bv, t)* M(SSx_{2n}, Bv, t)* M(Tv, ASx_{2n}, t) ≥ 1

letting n→∞,

F(M(z, Tv, Kt)* 1*,1* M(z, Tv, t)*1* M(z, Tv, t) *≥1.

Which implies by (F-2) that z=Tv, since Bv==Tv=z and B, T are compatible of type (α) we have TBv=BBv and so Tz=TBv=BBv=Bz. Thus from (iv), we have

F(M(Sx_{2n}, Tz, kt)* M(Ax_{2n}, Bz, t)* M(Sx_{2n}, Ax_{2n}, t)* M(Tz, Bz, t)* M(Sx_{2n}, Bz, t)* M(Tv, Ax_{2n}, t)) ≥ 1

Letting n→∞,

F(M(z, Tz, Kt) M(z, Tz, t)* 1*, 1* M(z, Tz, t)* M(z, Tz, t) *≥1.

Thus z=Tz=Bz. This means that z is the range of T and since T(x) ⊑ $A(X)$, there exists w belong sto X such that Aw=z.

Thus from (iv) we have for any t>0.

F(M(Sw, T, z, Kt)*M(Aw, Bz_1, t)* M(Sw, Aw, t)* M(Tz, Bz, t)* M(Sw, Bz, t) * M(Tz, Aw, t) ≥1.

F(M(Sw, z, Kt)*1* M(Sw, z, t)* 1* M(Sw, z, t) * 1) ≥1

And by (F-2) we have z=Sw =Aw =z and S, A, are compatible of type (α) we have z=Sz+Saw=AAw=Az and thus z=Az. consequently, z is a common fixed point S, T, A and B. The same results holds if we assume that T is continuous instead of S.

Finally we show that the point z is unique common fixed point of S, T, A, B. Suppose that S, T, A and B have another common fixed point $z_1$theny, by (iv) we have, for any t>0.

F(M(Sz, Tz_1, kt)* M(Az, Bz_1, t)* M(Sz, Az, t)* M(Tz_1, B$z_1 t$)* M(Sz, Bz_1, t)* M(Tz_1, Az, t)) ≥ 1

F(M(z,, z_1,Kt)* (M(z,, z_1,t) 1*,1* M(z, z_1,t)*1* M(z, z_1, t) *≥1

Thus from (F-3) we have (M (z,, $z_1 K$, t) ≥ (M(z,, z_1,t)we have z=z_1. This completes the proof.

Corollary 3.2: Let (X, M*) be complete fuzzy metric space and let A, B, S and T be self mappings on X such that the following conditions are satisfied (i)-(iii) and theorem 3.1. there exist k∈(0,1) such that for ever x, y∈ X and t>0

F(M(SX,TY,Kt)*M(AX,BY,t)*(M(SX,AX,t)*M(TY,NY,t)*M(SX,BY,t)*M(TY,AX,t) ≥ 1.

Then A, B, S and T have a unique common fixed point in X.

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