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# Certain Units in $M_2(Z)$ which are Uniquely Clean

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***ABSTRACT***

*An element of a ring is clean if it is the sum of an idempotent and a unit and it is uniquely clean if this representation is unique. Characterization for clean elements in* $M_2(Z)$ *has been given [2]. Using this characterization some sufficient conditions have been given for certain units to be uniquely clean.*

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## 1. INTRODUCTION:

An element of a ring is clean if it is the sum of a unit and an idempotent and if this representation is unique it is said to be uniquely clean. In the past few years, several attempts have been made to study the uniquely clean rings for ex. Nicholson and Zhou [1] have investigated the uniquely clean rings and found that these rings are closely related to Boolean rings and that every homomorphic image of a uniquely clean ring is uniquely clean.

However element wise characterizations are available nowhere. The only element wise result is that in any ring, the central idempotents are uniquely clean [1].

Recall that a matrix *A* in $M_2(Z)$ is clean if it is the sum of an idempotent matrix *E* (i.e., $E^2 = E$) and a unit *U* in $M_2(Z)$. In $M_2(Z)$, we have the following choices for *E*:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}$ with $yw = x - x^2$. In [2] we called *A* 1-clean if it is clean with $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ else, we called *A* 0-clean. One can see that a unit $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $M_2(Z)$ is always 0-clean with *E* = 0. In this paper we try to find some conditions under which *U* is uniquely clean, i.e., neither 1-clean nor 0-clean with $E = \begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}$ (with $yw = x - x^2$). We will begin with checking 0-cleanness of *U* with this choice of *E* and throughout the paper 0-cleanness will always be referred with respect to this E only.

For this choice of *E*, 0-cleanness of *U*, i.e., the condition that *U* - *E* is a unit, involves solving the following Diophantine equations:

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$$(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (ad - bc - a - 1)y = 0$$
$$(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (ad - bc - a + 1)y = 0$$

For the general diophantine equation $Ax^2 + Bxy + Cy^2 + Dx + Fy + G = 0$ we have the following cases:

(1) Linear Case: $A = B = C = 0$.
(2) Simple Hyperbolic Case: $A = C = 0$; $B \neq 0$.
(3) Parabolic Case: $B^2 - 4AC = 0$.
(4) Elliptic Case: $B^2 - 4AC < 0$
(5) Hyperbolic Case: $B^2 - 4AC > 0$

We shall say $U$ falls under one of the above categories, if the corresponding condition holds with $A = -b$, $B = a - d$, $C = c$, $D = b$, $F = (ad - bc - a - 1)$ or $(ad - bc - a + 1)$ and $G = 0$. Using the results of [2], we have shown that if $U$ falls in linear and simple hyperbolic case then it is not 0-clean. If $U$ falls in the parabolic case then a sufficient condition has been given under which it is not 0-clean. Further it will also be seen that the conditions for $U$ to fall in linear, simple hyperbolic and parabolic cases force U not to be 1-clean (for conditions of 1-cleanness see Theorem 3.1, 2). Hence we get a class of uniquely clean units in $M_2(Z)$ .

## 2. MAIN RESULTS:

Consider a unit matrix $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$ with $det(U) = \pm 1$. As mentioned in the introduction, we begin with the checking of 0-cleanness of $U$, by taking $E = \begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}$ (with $yw = x - x^2$) which falls under linear, simple hyperbolic and parabolic case.

We shall need the following theorems given in [2] (with the same notations and theorem numbers as in [2])

**Theorem: 3.1** A is 1-clean if and only if det(A) - Tr(A) = 0 or -2

**Theorem: 3.3** Let $E$ be an idempotent matrix with determinant zero and $E \notin X$ . Then $A - E$ is a unit $\Leftrightarrow$ one of the following Diophantine equations

$$(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (ad - bc - a - 1)y = 0$$
$$(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (ad - bc - a + 1)y = 0$$

has a solution $(x, y)$ with $x \neq 0,1$, $y \neq 0$ and $y \mid x(1-x)$

**Theorem: 3.4** $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 0-clean if and only if one of the following conditions is satisfied:

(i) A is a unit.
(ii) $ad - bc - d + bp = \pm 1$ for some $p$.
(iii) $ad - bc - a + bp = \pm 1$ for some $p$.
(iv)The diophantine equation $(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (ad - bc - a - 1)y = 0$ has a non- trivial solution.
(v)The diophantine equation $(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (ad - bc - a + 1)y = 0$ has a non-trivial solution.

 

**1. Linear Case:** In this case -*b* = 0, *a* - *d* = 0 and *c* = 0 i. e., *b* = 0, *a* = *d* and *c* = 0. If det(U)= 1, then we get *ad* = 1, which implies either *a* = *d* = 1or *a* = *d* = -1. Hence in this case we have $U=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ or $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$. Suppose $U=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then being a central idempotent it is uniquely clean (see [1]).

Next, we take $U=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$. One can see that in view of Theorem 3.4 (ii), (iii), (iv) and (v), *U* is not 0-clean.

Also as $det(U)\text{-}trace(U)\neq 0,\text{-}2$, in view of Theorem 3.1, *U* is not 1-clean.

If det (*U*) = -1 then we get *ad* = -1 which implies that *a* = 1, *d* = -1 or *a* = -1, *d* = 1 both of which contradict *a* = *d*. Thus no unit *U* with determinant -1 falls in this case. Hence units falling in the linear case are uniquely clean.

**2. Simple Hyperbolic Case:** In this case -*b* = 0, *c* = 0 and $a-d\neq 0$, i. e., *b* = 0, *c* = 0 and $a\neq d$. If det(*U*)= 1 then we get *ad* = 1 which implies that *a* = 1, *d* = 1 or *a* = -1, *d* = -1 both of which contradict $a\neq d$. Thus no unit with determinant 1 falls in this case.

If det(*U*)= -1 then we get *ad* = -1 which implies that *a* = 1, *d* = -1 or *a* = -1, *d* = 1. Hence in this case we have

$U=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ or $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$. Again, in view of Theorem 3.4 (ii), (iii), (iv) and (v), one can see that $U=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ and $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ both are not 0-clean.

For both of these *U*, one can clearly see that $det(U)\text{-}trace(U)\neq 0,\text{-}2$. Hence, by Theorem 3.1, both of them are not 1-clean.

Thus units falling in the simple hyperbolic case are also uniquely clean.

**3. Parabolic Case:** If det(*U*) = 1 then $B^2-4AC=0$ or $(a-d)^2-4(-b)(c)=0$ implies that $(a+d)^2-4(ad-bc)=0$ or $(a+d)^2-4=0$ or $a+d=\pm 2$.

If det(*U*) = -1 then $B^2-4AC=0$ or $(a-d)^2-4(-b)(c)=0$ implies that $(a+d)^2+4=0$ or $(a+d)^2=-4$ which is not possible for any integers *a* and *d*. Hence no unit with determinant -1 falls in this case.

In the discussion that follows we shall consider units falling in the parabolic case, i.e., units with determinant 1 and trace equal to $\pm 2$.

Note that for a unit *U* with det(*U*) = 1 and $trace(U)=\pm 2$, $det(U)\text{-}trace(U)\neq 0,\text{-}2$. Hence in view of Theorem 3.1, *U* cannot be 1-clean. In other words, a unit falling in parabolic case cannot be 1-clean. Thus for unique cleanness, we only require to concentrate on the 0-cleanness of these units.

In [2], we gave separate criteria for 0-cleanness of a matrix with

$E=\begin{bmatrix}1 & 0\\p & 0\end{bmatrix},\begin{bmatrix}0 & 0\\p & 1\end{bmatrix},\begin{bmatrix}1 & p\\0 & 0\end{bmatrix},\begin{bmatrix}0 & p\\0 & 1\end{bmatrix}$ which are special cases of $\begin{bmatrix}x & y\\w & 1-x\end{bmatrix}$ (with $yw=x-x^2$) (see Observation 3.2, [2]).

 

For a unit matrix $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$ with determinant 1, these necessary and sufficient criteria given in Observation 3.4, [1], will take the following form which we shall use in further discussion:

(i) If $E = \begin{bmatrix} 1 & 0 \\ p & 0 \end{bmatrix}$, then *U* - *E* is a unit iff $bp = d$ or *d* - 2.

(ii) If $E = \begin{bmatrix} 0 & 0 \\ p & 1 \end{bmatrix}$, then *U* - *E* is a unit iff $bp = a$ or *a* - 2.

(iii) If $E = \begin{bmatrix} 1 & p \\ 0 & 0 \end{bmatrix}$, then *U* - *E* is a unit iff $cp = d$ or *d* - 2.

(iv) If $E = \begin{bmatrix} 0 & p \\ 0 & 1 \end{bmatrix}$, then U-E is a unit iff $cp = a$ or *a* - 2.

Also, the Diophantine equations in Theorem 3.3 take the form:

$$(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (-a)y = 0 \quad (1)$$
$$(-b)x^2 + (a - d)xy + (c)y^2 + (b)x + (2 - a)y = 0 \quad (2)$$

The following two lemmas will be required in the proof of Theorem 2.3.

**Lemma: 2.1** Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a unit matrix in $M_2(Z)$ with det(*U*) = 1 and *a* + *d* = 2. If *a* = 0, 2 then *U* is not uniquely clean.

**Proof:** It is given that *a* + *d* = 2, hence *d* = 2 - *a*.

Suppose *a* = 0, then *d* = 2. As det(*U*) = 1, we get the following choices for *U*:

$$U = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \text{or} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

For both of these *bp* = *d* - 2 is satisfied if we take *p* = 0. Hence both are 0-clean in view of (i) and therefore not uniquely clean.

Suppose *a* = 2, then *d* = 0. Again as det(*U*) = 1, we get the following choices for U :

$$U = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \text{or} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

For both of these *bp* = *d* is satisfied if we take *p* = 0. Hence both are 0-clean in view of (i) and therefore not uniquely clean.

**Lemma: 2.2** Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a unit matrix in $M_2(Z)$ with det(*U*) = 1 and *a* + *d* = -2. If *a* = 0, 2, -2, -4 then *U* is not uniquely clean.

**Proof:** As *a* + *d* = -2, *d* = -2 - *a*.

If *a* = 0, then *d* = -2. As det(*U*) = 1, following are the possibilities

 

$$U = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \text{or} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

For the first one $bp = d$ is satisfied if we take $p = 2$ and for the second one it is satisfied if we take $p = -2$. Thus both of these are 0-clean in view of (i) and hence they are not uniquely clean.

If $a = 2$ then $d = -4$. As det($U$) = 1, we have the following possibilities

$$U = \begin{bmatrix} 2 & -9 \\ 1 & -4 \end{bmatrix}, \begin{bmatrix} 2 & 9 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -9 & -4 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 9 & -4 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

For the first and second forms, $cp = d$ is satisfied by taking $p = -4$ and $p = 4$ respectively. Thus the first two are 0-clean in view of (iii) and hence are not uniquely clean. The third and fourth forms, being transposes of the first and second forms respectively, are also not uniquely clean.

For the fifth and sixth forms, $bp = d - 2$ is satisfied if we take $p = 2$ and $p = -2$ respectively. Thus they are 0-clean in view of (i) and hence they are not uniquely clean.

If suppose $a = -2$ then $d = 0$. Again as det($U$) = 1, we have the following possibilities

$$U = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \text{or} \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$$

As $bp = d$ is satisfied for both if we take $p = 0$, in view of (i) both are 0-clean and hence are not uniquely clean.

If $a = -4$ then $d = 2$. Since det($U$) = 1, we get the following choices for U:

$$U = \begin{bmatrix} -4 & -9 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -4 & 9 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -4 & 1 \\ -9 & 2 \end{bmatrix}, \begin{bmatrix} -4 & -1 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix}$$

For the first and second forms $cp = d$ is satisfied if we take $p = 2$ and p = -2 respectively. Thus both are 0-clean in view of (iii) and hence they are not uniquely clean. The third and fourth forms are also not uniquely clean being the transposes of first and second forms respectively.

Finally, one can see that for the fifth and sixth forms $bp = d - 2$ is satisfied for $p = 0$. Hence in view of (i) they are 0-clean and hence not uniquely clean. This completes the proof.

In what follows we shall state and prove a sufficient condition for a unit which falls in parabolic case to be uniquely clean. We will use the following method given by Dario Alpern to solve equation of the form

$Ax^2 + Bxy + Cy^2 + Dx + Fy + G = 0$ in the Parabolic Case, i.e., when $B^2 - 4AC = 0$ which is available on net.

Let $g = gcd(A, C)$, $P = \dfrac{A}{G} \geq 0$, $Q = \dfrac{B}{G}$, $R = \dfrac{C}{G} \geq 0$.

Since $Q^2 = 4PR$ is positive, we can choose $g$ with the same sign of $A$. In this way $P$ and $R$ will be positive (or one of them zero).

The expression $Q^2 - 4PR = 0$ implies that $\dfrac{Q^2}{4} = PR$. Since $gcd(P, R) = 1$, both $P$ and $R$ are perfect squares.

Multiplying the original equation by $\sqrt{P}$ :

 

$$\sqrt{P}g(Px^2 + Qxy + Ry^2) + \sqrt{P}Dx + \sqrt{P}Fy + \sqrt{P}G = 0$$

$$\sqrt{P}g(\sqrt{P}x + \sqrt{R}y)^2 + \sqrt{P}Dx + \sqrt{P}Fy + \sqrt{P}G = 0$$

where for $\sqrt{R}$ , the sign of is taken $\frac{B}{A}$

Adding and subtracting $\sqrt{R}Dy$ :

$$\sqrt{P}g(\sqrt{P}x + \sqrt{R}y)^2 + D(\sqrt{P}x + \sqrt{R}y) - \sqrt{R}Dy + \sqrt{P}Fy + \sqrt{P}G = 0$$

Let u = $\sqrt{P}x + \sqrt{R}y$ (3)

$$\sqrt{P}gu^2 + Du + (\sqrt{P}F - \sqrt{R}D)y + \sqrt{P}G = 0$$

or

$(\sqrt{R}D - \sqrt{P}F)y = \sqrt{P}gu^2 + Du + \sqrt{P}G$ (4)

There are two cases:

$$\sqrt{R}D - \sqrt{P}F = 0 \qquad \text{(two parallel lines)}$$

or

$$\sqrt{R}D - \sqrt{P}F \neq 0 \qquad \text{(a parabola)}$$

In the first case, $\sqrt{R}D - \sqrt{P}F = 0$ . Then from eq.(4) we get $\sqrt{P}gu^2 + Du + \sqrt{P}G = 0$ .

Since *x* and *y* should be integers, the equation (3) implies that the number *u* (the root of the above equation) should be also be an integer. Let $u_1$ and $u_2$ be the two roots of the above equation.

From (3) we have

$$\sqrt{P}x + \sqrt{R}y - u_1 = 0 \text{ and } \sqrt{P}x + \sqrt{R}y - u_2 = 0$$

which can be solved with the methods for linear equation.

In the second case, $\sqrt{P}gu^2 + Du + \sqrt{P}G$ should be a multiple of $\sqrt{R}D - \sqrt{P}F$ .

Let $u_0$ , $u_1$ ....... be the values of *u* in the range $0 \le u < |\sqrt{R}D - \sqrt{P}F|$ for which the above condition holds.

So,

$u = u_i + (\sqrt{R}D - \sqrt{P}F)t$ , where *t* is any integer (5)

Replacing (5) in (4) we get

$$(\sqrt{R}D - \sqrt{P}F)y = \sqrt{P}g[u_i + (\sqrt{R}D - \sqrt{P}F)t]^2 + D[u_i + (\sqrt{R}D - \sqrt{P}F)t] + \sqrt{P}G$$

or

$$y = \sqrt{P}g(\sqrt{R}D - \sqrt{P}F)t^2 + (D + 2\sqrt{P}gu_i)t + \frac{\sqrt{P}gu_i^2 + Du_i + \sqrt{P}G}{(\sqrt{R}D - \sqrt{P}F)}$$

 

From (3) and (5)

$$u = \sqrt{P}x + \sqrt{R}y = u_i + (\sqrt{R}D - \sqrt{P}F)t$$

$$\sqrt{P}x = (\sqrt{P}\sqrt{R})g[\sqrt{P}F - \sqrt{R}D]t^2 + (\sqrt{R}D - \sqrt{P}F - 2\sqrt{P}\sqrt{R}gu_i - \sqrt{R}D)t + u_i - \sqrt{R}\left[\frac{\sqrt{P}gu_i^2 + Du_i + \sqrt{P}G}{(\sqrt{R}D - \sqrt{P}F)}\right]$$

or

$$x = \sqrt{R}g(\sqrt{P}F - \sqrt{R}D)t^2 - (F + 2\sqrt{R}gu_i)t - \left[\frac{\sqrt{R}gu_i^2 + Fu_i + \sqrt{R}G}{(\sqrt{R}D - \sqrt{P}F)}\right]$$

**Theorem: 2.3**. Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a unit in $M_2(Z)$ with *det*(*U*) = 1. For *a* + *d* = 2, if $a \neq 0,2$ and *b* = -*c* then *U* is uniquely clean. For *a* + *d* = -2, if $a \neq 0,2,-2,-4$ and *b* = -*c* then *U* is uniquely clean.

**Proof: Case 1:** *a* + *d* = 2

Applying the conditions *b* = -*c*, *a* + *d* = 2 and det(*U*) = 1, we note that *U* is of the form :

$$U = \begin{bmatrix} a & (a-1) \\ -(a-1) & 2-a \end{bmatrix} \text{ or } \begin{bmatrix} a & -(a-1) \\ (a-1) & 2-a \end{bmatrix}$$

Consider the first form $U = \begin{bmatrix} a & (a-1) \\ -(a-1) & 2-a \end{bmatrix}$

For *a* = 1, we get $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is uniquely clean being a central idempotent. Thus the theorem holds for *a* = 1. Hence in the discussion that follows, we let $a \neq 1$.

As $a \neq 0$ or 2 and (*a* – 1) cannot divide *a* or *a* - 2, we cannot find an integer *p* for which the conditions (i) to (iv) are satisfied for *U*.

For *U*, the Diophantine equations (1) and (2) take the form:

$$[-(a-1)]x^2 + (2a-2)xy + [-(a-1)]y^2 + (a-1)x + (-a)y = 0 \quad (6)$$

$$[-(a-1)]x^2 + (2a-2)xy + [-(a-1)]y^2 + (a-1)x + (2-a)y = 0 \quad (7)$$

Applying the method discussed before this theorem, we solve the above equations as follows:

Consider equation (6)

$$[-(a-1)]x^2 + (2a-2)xy + [-(a-1)]y^2 + (a-1)x + (-a)y = 0$$

or

$$(a-1)x^2 + (2-2a)xy + (a-1)y^2 + [-(a-1)]x + ay = 0$$

Let

$$g = gcd\,(a-1, a-1) = a-1$$

 

and let

$$P=\frac{A}{g}=\frac{a-1}{a-1}=1$$

$$Q=\frac{B}{g}=\frac{2-2a}{a-1}=-2$$

$$R=\frac{C}{g}=\frac{a-1}{a-1}=1$$

Then $\sqrt{P}=1,\ \sqrt{R}=1$. Now $\frac{B}{A}=\frac{2-2a}{a-1}=-2$ is negative, so according to the method we shall give the sign of $\frac{B}{A}$ to $\sqrt{R}$. Hence we take $\sqrt{P}=1,\ \sqrt{R}=-1$. One can see that $\sqrt{R}D-\sqrt{P}F=-1$. Hence we shall take all integers $u$ such that

$$0\le u<|\sqrt{R}D-\sqrt{P}F|$$

or

$$0\le u<1$$

Only one such value of $u$ is possible namely $u = 0$. For this value of $u$, solutions for $x$ and $y$ as specified by the method are

$$x=-(a-1)t^2-at,\quad y=-(a-1)t^2-(a-1)t=-(a-1)t(t+1)$$

where we choose any $t$ such that $x\neq 0, 1$ and $y\neq 0$ (i. e., $t\neq 0,-1$).
Consider

$$\begin{aligned}x(1-x) &= -[(a-1)t^2+at][(1+(a-1)t^2+at)\\ &= -[(a-1)^2t^4+2a(a-1)t^3+(a-1+a^2)t^2+at]\\ &= (-t)[(a-1)^2t^3+2a(a-1)t^2+(a-1+a^2)t+a]\\ &= (-t)[(a-1)^2t^2(t+1)+(a^2-1)t(t+1)+a(t+1)]\\ &= (-t)(t+1)[(a-1)^2t^2+(a^2-1)t+a]\end{aligned}$$

As $(a - 1)$ cannot divide the expression $[(a-1)^2t^2+(a^2-1)t+a]$ $(a\neq 0, 2)$, $y$ cannot divide $x(1-x)$.

Next, consider the equation (7)

$$[-(a-1)]x^2+(2a-2)xy+[-(a-1)]y^2+(a-1)x+(2-a)y=0$$

or

$$(a-1)x^2+(2-2a)xy+(a-1)y^2+[-(a-1)]x+(a-2)y=0$$

Following the same method, we have

$$g = gcd(a-1, a-1) = a-1$$

$$P=\frac{A}{g}=\frac{a-1}{a-1}=1$$

$$Q=\frac{B}{g}=\frac{2-2a}{a-1}=-2$$

$$R=\frac{C}{g}=\frac{a-1}{a-1}=1$$

$\sqrt{P}=1$ and we take $\sqrt{R}=-1$ (giving it the sign of $\dfrac{B}{A}=\dfrac{2-2a}{a-1}=-2$ which is negative). As $\sqrt{R}D-\sqrt{P}F=1$, we shall take all integers $u$ such that

$$0 \le u < |\sqrt{R}D-\sqrt{P}F|$$

or

$$0 \le u < 1$$

As $u$ is an integer, there is only one choice for $u$ namely $u$ = 0. Using the formulae for $x$ and $y$, for this choice of $u$ we get

$$x=(a-1)t^2-(a-2)t\ ,\ y=(a-1)t^2-(a-1)t=(a-1)t(t-1)$$

where we choose any $t$ such that $x \neq 0,1$ and $y \neq 0$ (i.e., $t \neq 0,1$).

Consider

$$\begin{aligned} x(1-x) &= [(a-1)t^2-(a-2)t][1-(a-1)t^2+(a-2)t] \\ &= [-(a-1)^2t^4+2(a-1)(a-2)t^3+[(a-1)-(a-2)^2]t^2-(a-2)t] \\ &= t[-(a-1)^2t^3+2(a-1)(a-2)t^2+[(a-1)-(a-2)^2]t-(a-2)] \\ &= t[-(a-1)^2t^2(t-1)+(a-1)(a-3)t(t-1)+(a-2)(t-1)] \\ &= t(t-1)[-(a-1)^2t^2+(a-1)(a-3)t+(a-2)] \end{aligned}$$

As one can see that ($a$ - 1) cannot divide $[-(a-1)^2t^2+(a-1)(a-3)t+(a-2)]$ ($a \neq 0,2$), $y$ cannot divide $x$(1 - $x$).

Thus neither equation (6) nor equation (7) yield a solution ($x$, $y$), $x \neq 0,1$ and $y \neq 0$ such that $y \mid x(1-x)$. Hence $U$ does not satisfy the condition in Theorem 3.3.

Thus from all the above discussion it follows that $U$ is not 0-clean and hence uniquely clean.

Note that the other form $\begin{bmatrix} a & -(a-1) \\ (a-1) & 2-a \end{bmatrix}$ of $U$ is simply the transpose of $\begin{bmatrix} a & (a-1) \\ -(a-1) & 2-a \end{bmatrix}$. Hence it is also uniquely clean.

**Case 2:** $a+d=-2$

Applying the conditions $b=-c$, $a+d=-2$ and $det(U)=1$, $U$ takes the form :

$$U=\begin{bmatrix} a & (a+1) \\ -(a+1) & -2-a \end{bmatrix} \text{ or } \begin{bmatrix} a & -(a+1) \\ (a+1) & -2-a \end{bmatrix}$$

Consider the first form $U=\begin{bmatrix} a & (a+1) \\ -(a+1) & -2-a \end{bmatrix}$

If $a$ = -1 then $U=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ which we know is uniquely clean by the linear case. Thus for $a$ = -1 the theorem is true.

Hence for further discussion we let $a \neq -1$.

Note that as $a \neq 0, 2, -2, -4$, (i) to (iv) cannot be satisfied for *U* with *p* = 0. It is not possible that (*a* + 1) divides *a* ($a \neq 0, -2$). Similarly (*a* + 1) cannot divide (-2 - *a*) ($a \neq 0, -2$). Hence there cannot exist any integer $p\,(\neq 0)$ for which *bp* = *a* or *d* and *cp* = *a* or *d* is satisfied for U. One can check that (*a* + 1) does not divide (*a* - 2) as follows:

Let *a* > 0. Then as (*a* + 1) > (*a* - 2) and $a \neq 2$, *a* + 1 cannot divide *a* - 2.

Let *a* < 0 ( $a = 0$ is not taken because of the sufficient condition). Let $m = a + 1$, n=a - 2 so that n - m= -3. As |m < |n| , there is a possibility that m|n . Suppose m|n . Then

$$\begin{aligned} & m \mid m - 3 \\ \Rightarrow\ & m \mid -3 \\ \Rightarrow\ & m = -1, -3 \quad (as\ as\ a < 0, m < 1) \\ \Rightarrow\ & a + 1 = -1, -3 \\ \Rightarrow\ & a = -2, -4 \end{aligned}$$

which is a contradiction to our assumption that $a \neq -2, -4$.

Hence (*a* + 1) cannot divide (*a* - 2) in this case also. From this we can conclude that there cannot exist any integer $p\,(\neq 0)$ such that *bp* = *a* - 2 or *cp* = *a* - 2 is satisfied for U. In view of the sufficient condition $a \neq 0, 2, -2, -4$, it can similarly be proved that (*a* + 1) does not divide -4 - *a* or -(*a* + 4) and therefore neither *bp* = *d* - 2 nor *cp* = *d* - 2 is satisfied for *U* for any integer $p\,(\neq 0)$. Thus *U* cannot fall in the conditions (i) to (iv).

For U , the diophantine equations (1) and (2) respectively take the following forms:

$$[-(a+1)]x^2 + (2a+2)xy + [-(a+1)]y^2 + (a+1)x + (-a)y = 0 \qquad (8)$$

$$[-(a+1)]x^2 + (2a+2)xy + [-(a+1)]y^2 + (a+1)x + (2-a)y = 0 \qquad (9)$$

Consider equation (8), which can be written as

$$(a+1)x^2 + \left[-(2a+2)\right]xy + (a+1)y^2 + \left[-(a+1)\right]x + ay = 0$$

Following the method specified before the theorem, we have

$$g = gcd(a+1, a+1) = a+1$$

$$P = \frac{A}{g} = \frac{a+1}{a+1} = 1$$

$$Q = \frac{B}{g} = \frac{-(2+2a)}{a+1} = -2$$

$$R = \frac{C}{g} = \frac{a+1}{a+1} = 1$$

$\sqrt{P} = 1$ and we take $\sqrt{R} = -1$ (giving it the sign of $\frac{B}{A} = \frac{-(2+2a)}{a+1} = -2$ which is negative).

The quantity $\sqrt{R}D - \sqrt{P}F = 1$, so we shall take all integers *u* such that

$$0 \leq u < |\sqrt{R}D - \sqrt{P}F|$$

or

$$0 \le u < 1$$

Clearly the only choice for $u$ is $u = 0$. For this choice of $u$, we get

$$x = (a+1)t^2 - at \;\; and \;\; y = (a+1)t(t-1)$$

where we choose any $t$ such that $x \neq 0,1$ and $y \neq 0$ (i.e., $t \neq 0,1$)

Consider

$$\begin{aligned} x(1-x) &= \left[(a+1)t^2 - at\right]\left[1-(a+1)t^2+at\right] \\ &= \left[-(a+1)^2 t^4 + 2a(a+1)t^3 + \left[(a+1)-a^2\right]t^2 - at\right] \\ &= t\left[-(a+1)^2 t^3 + 2a(a+1)t^2 + \left[(a+1)-a^2\right]t - a\right] \\ &= t\left[-(a+1)^2 t^2 (t-1) + (a^2-1)t(t-1) + a(t-1)\right] \\ &= t(t-1)\left[-(a+1)^2 t^2 + (a^2-1)t + a\right] \end{aligned}$$

As $(a + 1)$ cannot divide $\left[-(a+1)^2 t^2 + (a^2-1)t + a\right]$ ($a \neq 0, -2$), y cannot divide $x(1 - x)$. Next, consider the equation (9) which can be written as

$$(a+1)x^2 + \left[-(2a+2)\right]xy + (a+1)y^2 + \left[-(a+1)\right]x + (a-2)y = 0$$

For this equation we have

$$g = a + 1, P = 1, R = 1, \sqrt{P} = 1 \text{ and } \sqrt{R} = -1$$

The quantity $\sqrt{R}D - \sqrt{P}F = 3$, so we shall take all integers $u$ such that

$$0 \le u < |\sqrt{R}D - \sqrt{P}F|$$

or

$$0 \le u < 3$$

We get three choices of $u$ which are $u_0 = 0, u_1 = 1, u_2 = 2$.

For $u_0 = 0$ we get $x = 3(a+1)t^2 - (a-2)t$ and $y = (a+1)t(3t-1)$ where we choose any $t$ such that $x \neq 0,1$ and $y \neq 0$ (i.e., $t \neq 0$).

Consider

$$\begin{aligned} x(1-x) &= \left[3(a+1)t^2 - (a-2)t\right]\left[1-3(a+1)t^2+(a-2)t\right] \\ &= \left[-9(a+1)^2 t^4 + 6(a+1)(a-2)t^3 + \left[3(a+1)-(a-2)^2\right]t^2 - (a-2)t\right] \\ &= t\left[-9(a+1)^2 t^3 + 6(a+1)(a-2)t^2 + \left[3(a+1)-(a-2)^2\right]t - (a-2)\right] \\ &= t\left[-3(a+1)^2 t^2 (3t-1) + (a+1)(a-5)t(3t-1) + (a-2)(3t-1)\right] \\ &= t(3t-1)\left[-3(a+1)^2 t^2 + (a+1)(a-5)t + (a-2)\right] \end{aligned}$$

 

As we have seen earlier that $(a+1)$ cannot divide $(a-2)$, y cannot divide $x(1-x)$ in this case.

For $u_1 = 1$ we get

$$x = 3(a+1)t^2 + (a+4)t + 1 \text{ and } y = (a+1)t(3t+1)$$

where we choose any $t$ such that $x \neq 0, 1$ and $y \neq 0$ (i.e., $t \neq 0$).

Consider

$$\begin{aligned} x(1-x) &= \left[3(a+1)t^2 + (a+4)t+1\right]\left[-3(a+1)t^2 - (a+4)t\right] \\ &= \left[-9(a+1)^2 t^4 - 6(a+1)(a+4)t^3 - \left[(a+4)^2 + 3(a+1)\right]t^2 - (a+4)t\right] \\ &= (-t)\left[9(a+1)^2 t^3 + 6(a+1)(a+4)t^2 + \left[(a+4)^2 + 3(a+1)\right]t + (a+4)\right] \\ &= (-t)\left[3(a+1)^2 t^2(3t+1) + (a+1)(a+7)t(3t+1) + (a+4)(3t+1)\right] \\ &= (-t)(3t+1)\left[3(a+1)^2 t^2 + (a+1)(a+7)t + (a+4)\right] \end{aligned}$$

As we have seen earlier that $(a+1)$ cannot divide $(a+4)$, y cannot divide $x(1-x)$ in this case also.

For $u_2 = 2$ we get

$$x = 3(a+1)t^2 + (3a+6)t + \frac{(2a+8)}{3} \text{ and } y = 3(a+1)t^2 + (3a+3)t + \frac{(2a+2)}{3}$$

where we choose any $t$ such that $x \neq 0, 1$ and $y \neq 0$.

For $x$ and $y$ to be integers we must have

$$3 \mid 2(a+4) \text{ and } 3 \mid 2(a+1)$$

or

$$3 \mid (a+4) \text{ and } 3 \mid (a+1)$$

for which it is sufficient that

$$3 \mid (a+1)$$

as $(a+1)$ and $(a+4)$ differ by 3.

For those $U$ in which 3 does not divide $(a+1)$, $x$ and $y$ will not be integers. Hence $U$ will not be 0-clean. Suppose for $U$ it is seen that $3 \mid (a+1)$. Then we can write $a+1 = 3k$, where $k \neq 0, 1, -1$ as $a \neq -1, 2, -4$ respectively. Hence values of $x$ and $y$ will become:

$$\begin{aligned} x &= 3(3k)t^2 + 3(3k+1)t + 2(k+1) \\ &= 9kt^2 + (9k+3)t + 2(k+1) \\ &= \left(9kt^2 + 9kt + 2k\right) + 3t + 2 \end{aligned}$$

and

$$\begin{aligned} y &= 3(3k)t^2 + 3(3k)t + 2k \\ &= 9kt^2 + 9kt + 2k \\ &= k\left(9t^2 + 9t + 2\right) \end{aligned}$$

 

Therefore we can write

$$x = y + 3t + 2$$

Consider

$$\begin{aligned} x(1-x) &= (y + 3t+2)[1-(y + 3t+2)] \\ &= (y + 3t+2)(-y-3t-1) \\ &= -y^2 - y(3t+1) - y(3t+2) - (3t+2)(3t+1) \\ &= -y^2 - y(3t+1) - y(3t+2) - (9t^2+9t+2) \\ &= y\left[-y-(3t+1)-(3t+2)\right] - (9t^2+9t+2) \end{aligned}$$

As $k \neq 1 \text{or} -1$, y cannot divide $x(1 - x)$.

Thus in all the above cases one can see that U does not satisfy the condition in Theorem 3.3. Hence U is not 0-clean and therefore uniquely clean.

Note that as $\begin{bmatrix} a & -(a+1) \\ (a+1) & -2-a \end{bmatrix}$ is the transpose of $\begin{bmatrix} a & (a+1) \\ -(a+1) & -2-a \end{bmatrix}$, it is also uniquely clean under the same sufficient conditions. This completes the proof.

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