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S-SPECIAL DEFINITE RINGS AND S-SPECIAL DEFINITE FIELDS

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ABSTRACT

In this paper, we study Smarandache (S) specialdefinite rings and Smarandache (S)specialdefinite fields. We givecharacterizations of a S-special definite ringanda S-special definite field and determine some properties of each of them and obtain some result.

Keywords: *S - special definite ring,S - special definite field.*

2010 Mathematics subject classification: *12E99, 13A99.*

INTRODUCTION

Smarandachealgebraic structures introduced by Raul Padilla and Florentine Smarandache[1] and [2]. S-special definite algebraic structures suchas S - special definite groups,S - special definite rings and S - special definite fields definedby W.B.Vasantha Kandasamy[3]. These new structures are defined as those strong algebraic structures which contain weak algebraic structures. For instance, the existence of a semigroup in a group or a ring in a field or a semiring in a ring.In this work westudy S-special definite rings and S - special definite fields. This paper consists of threesections.In section one we state basic definitions on Smarandache algebraic structures that we need in our work. In sectiontwowe givea characterization of S - special definite rings. It is shown that every S - special definite ring has characteristic zero and that every ring of characteristic zero with identity is a S - special definite ring.Acharacterization of a S - special definite ringis given usingits S - special definite substructures. A condition is given under which every non trivial subring of a S - special definite ring is a S - special definite ring.In section three characterization of S-special definite fields is given. It is shown that If F is a S-special definite field, then F containsan infinite countable numberof subrings which are not field.We show that a finite field can not beS -special definite field. Moreoverwe study S-definite special fields and we show that a field F is a S-definite special fieldif and only if F is a field of characteristic zero.

1. BACKGROUND

In thissection we state basic definitions on S-algebraic structures thatwe needin our work.

Theorem 1.1[4, P.50]: A finite semigroup is a group if and only if it is satisfies the cancellation law.

Theorem 1.2 [5, P.172]: If R is a finiteringwith more than one element withno divisor of zero, then R is a field.

Theorem 1.3 [4,P.**249]:** Let R be a ring with more than one element such that x R = R, for everynon zero element $x \in R$. Then R is a division ring.

Definition 1.4: [6] (S, +, *) is called a semiring, if it satisfies the following conditions
1. (S, +) is a commutative semigroup with identity.
2. (S, *) is a semigroup.
3. (a + b) * c = a* c + b* c and c* (a + b) = c * a + c * b, for all a, b, c in S.

Definition 1.5: [3**, P.61**] A ring R is said to be S -special definite ring if there is a non empty subset S of R such that S is just a semiring (S is a semiring under the induced operations of R, but not a ring). If H itself is a S-special definite ring, then H is called a S-special definite subring of R.

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Definition 1.6 [7, P.38]: A S - ring R is a ring such that a proper subset F of R is a field with respect to the induced operations of R.

Definition 1.7 [3, P.50]: A field F is said to be S - special definite field if there is anon empty subset R of F such that R is just a ring (R is a ring under the induced operation of F but not a field). If H itself is a S-special definite field, then we call H a S- special definite subfield of F. *If F has no proper S-special definite subfield then we call F to be a S - special definite prime field.*

Definition1.8 [6]: Let S be a non empty set. Then S is said to be a semifield, if it satisfies thefollowing conditions

1. S is a commutative semiring with 1.
2. S is a strict semiring, that is if a + b = 0, then a= b = 0, for all a, b in S.
3. If a b = 0, then either a = 0 or b = 0, for all a, b in S.

Definition 1.9 [3, P.75]: Let F be a field and A a proper subset of F which is a semifield under the operations of F. Then we say F is a S - definite special field.

2. S -SPECIAL DEFINITE RINGS

In this section we givea characterization of a S - special definite ring. It is shown that every S-special definite ring has characteristic zero and that every ring of characteristic zero with identity is a S- special definite ring.A characterization of a S - special definite ring is given using its S - special definite substructures. We give a condition under which every non trivial subring of a S - special definitering is S-special definite subring.A necessary and sufficient condition is given for group rings, polynomial rings and ring of matrices to be S - special definite rings.

Theorem 2.1: Let R be a ring. Then R is a S-special definite ring if and only ifthere exists $a\in R$ such that $na\neq 0$, for all $n\in \mathbb{Z}^+$.

Proof: Suppose R is a S-special definite ring and let $S \subset R$ be just a semiring. Suppose for each $a\in S$ thereexists $n\in \mathbb{Z}^+$ such thatn a=0. But (n-1) $a\in$ S so, -a= (n-1) $a\in S$, which shows that S is a ring, which is acontradiction with assumption S is just a semiring.Thenthere exists $a\in S$ such that $na\neq 0$, for all $n\in \mathbb{Z}^+$. (R, +) containsan element of infinite order.

Conversely suppose that there exists $a\in$ R such that $na\neq 0$, for all $n\in \mathbb{Z}^+$.

Let $S=\{na+ba: n \in\mathbb{Z}^+ \cup\{0\} \text{and } b\in R \}$. Clearly S is a semiring.

If S is just a semiring, then the proofis complete, otherwise S is a ring and every element of S has an additive inverse in S. Take any such element say 2a, then thereexists an element $na+ba\in S$ such that 2a+ n a+ba=0, thus (2 + n) a+ba=0, thus

$$-b\, a = (2 + n)\, a \text{ and } -b\neq 0 \qquad (1)$$

Let $S^*= \{b\in R;\ ba=na \text{ for some } n\in\mathbb{Z}^+\} \cup \{0\}$. Then $-b\in S^*$. This means that $S^*\neq \emptyset$. We claim that S^* is just a semiring. If b_1,b_2 are two non zero elements in S^*, then $b_1 a=n_1 a, b_2 a=n_2 a$, for some $n_1, n_2\in\mathbb{Z}^+$, thus $(b_1+b_2) a=b_1 a +b_2 a= n_1 a+ n_2 a= (n_1 + n_2) a$, and $(b_1\ b_2) a=b_1 (b_2\ a)=b_1 n_2 a= n_2(b_1 a)=(n_1 n_2) a$, thus $b_1 +b_2\in S^*$ and $b_1 b_2\in S^*$.

If $b_1=0$ or $b_2=0$, then $b_1 b_2 =0\in S^*$ and ($b_1 +b_2= b_1\in S^*$ or $b_1 +b_2= b_2\in S^*$).

Then S^* is a semiring. $-b\in S^*$ and $-b$ has no additive inverse in S, since otherwise if there exists an element $b_1\in S^*$ such that $-b +b_1=0$, then since $b_1\in S^*$ and $b_1\neq 0$ by (if $b_1=0$, then $-b =0$, which is a contradiction), then $b_1 a=n_1 a$, for some

$$n_1\in \mathbb{Z}^+ \qquad (2)$$

$0=(-b+b_1) a= -ba+b_1a$ from (1) and (2) we get, $0=(2 + n) a + n_1 a =(2 + n+ n_1) a$, but $na\neq 0$, for all $n\in \mathbb{Z}^+$, then $2 + n+ n_1\leq 0$ which is a contradiction , with assumption ($n_1\in \mathbb{Z}^+$ and $n \in\mathbb{Z}^+ \cup\{0\}$), therefore$-b$ has no additive inverse in S, this means that $(S^*, +,.)$ is just a semiring, consequently R is a S-specialdefinite ring.

Examples 2.2:

1. For an infinite set X the ring $(P(X),\Delta,\cap)$ is not a S-specialdefinite ring.
2. $(Z_{p^\infty}, +,.)$ with trivial multiplication is an infinite ring of characteristic zero, but it is not a S-special definite ring, since for each $a\in Z_{p^\infty}$, there exists $n\in\mathbb{Z}^+$such that n.a=0.
3. $(\mathbb{Z},+,.)$ is a S-special definite ring , since it contains $(\mathbb{Z}^+,+,.)$, which is a semiring.

Corollary 2.3: Every S-special definite ring is of characteristic zero.

Proof: The proof is a direct consequence of Theorem 2.1. From Corollary 2.3, we deduce that a finite ring can not be a S-special definite ring.

The converse of Corollary2.3, is not true in general asthe infinite direct sum $\oplus Zp$, p runs over all prime numbersis a ring of characteristiczero, but it is not a S-special definite ring.

Proposition 2.4: Let R is a ring with identity element 1 of characteristic zero.Then R is a S-special definite ring.

Proof: Let$S=\{ n.1;\ n \in \mathbb{Z}^+\}\cup\{0\}$. Clearly S is a semiring. For each $n\in\mathbb{Z}^+$, n1 has no additive inverse in S, since if n 1+m 1=0, where $m \in\mathbb{Z}^+\cup\{0\}$, then $(n+m)1 =0$, consequently $(n+m)a=(n+m)(1a) = ((n+m)1)a =0$, for all $a\in R$, which is a contradiction with R is of characteristic zero. Then S is just a semiring, hence R is a S-special definite ring. The converse of Proposition 2.4, is not true in general as **(**$2\mathbb{Z}$, +,.) is a S-special definite ring, without identity.

In the following proposition a necessary and sufficient conditionis given under which the direct product of two rings is a S-special definite ring.

Proposition 2.5: Let R_1, R_2 are two rings . Then $R_1\times R_2$ is a S-special definite ring if andonly if at least one of R_1 or R_2 is S-special definite ring.

Proof: Suppose R_1 is a S-special definite ring. Then there exists $S\subset R$ such that (S, +,.) is just a semiring. Hence $S\times \{0\}$ is just asemiring of $R_1\times R_2$.So, $R_1\times R_2$ is S-special definite ring. The proof is similar when R_2 is S-special definite ring.

Conversely suppose that $R_1\times R_2$ is S-special definite ring. Then by Theorem 2.1, there exists $(a,b) \in R_1\times R_2$ such that (a,b) is of infinite order with respect to addition, thus $a \in R_1$ is of infinite order with respect to addition or $b\in R_2$ is of infinite order with respect to addition, since otherwise (there exist $n, m \in \mathbb{Z}^+$ such that n a=0 and mb=0, then nm (a,b) = (m(na), n(mb)) =(0, 0),which is a contradiction), then by Theorem 2.1, R_1 is S-special definite ring or R_2 is S-special definite ring. More generally we have

Corollary 2.6: If $R_1, R_2,\ldots, R_n$ are rings, then $R_1\times R_2\times\ldots\times R_n$ is a S-special definite ringif and only if at least one of $R_1, R_2,\ldots, R_n$ is a S-special definite ring.

Proposition 2.7: Every ring can be imbedded in a S-special definite ring.

Proof: Let R be a ring. Since ($\mathbb{Z}$, +,.) is a S-special definite ring, thenby Proposition 2.5, $R\times\mathbb{Z}$ is a S-special definite ring. But $R\times\{0\}$ is subring of $R\times\mathbb{Z}$ which is isomorphic to R. Then R is imbedded in $R\times\mathbb{Z}$.

Theorem 2.8: Let RG bethe group ring of the group G over the ring R. Then RG is a S-special definite ring if and only if R is a S-special definite ring.

Proof: Suppose that R is a S-special definite ring, thenby Theorem 2.1 there exists $a\in R$ such that $na\neq 0$, for all $n\in\mathbb{Z}^+$. Thenn $(ae_G) = (na)\, e_G\neq 0_{RG}$, for all $n\in\mathbb{Z}^+$, by Theorem2.1, RG is a S-special definite ring.

Conversely suppose that RG is a S-special definite ring. ByTheorem 2.1, there exists $a_0+a_1g_1+ a_2g_2+\ldots+a_ng_n\in RG$, where $a_0,\ldots,a_n\in R$ and $g_1,\ldots,g_n\in G$ such that $n(a_0+a_1g_1+\ldots+a_ng_n) \neq 0$, for all $n\in\mathbb{Z}^+$.Suppose that every element of (R,+) is of finite order, soeveryelement $a_i\in R$ there exists $m_i\in\mathbb{Z}^+$ such that $m_i\, a_i=0$, so $m_0m_1\ldots m_n(a_0+a_1g_1+a_2g_2+\ldots+a_ng_n) = 0$ which is a contradiction, so there exists $a\in R$ such that $n.a\neq 0$, for all $n\in\mathbb{Z}^+$, then R is a S-special definite ring.

Theorem 2.9: Let R be a ring. Then R[x] is a S-special definite ring if and only if R is a S-special definite ring.

Proof: Suppose that R is a S-special definite ring, thus there exists just a semiring S of R such that $S\subset R\subset R[x]$, so R[x] is a S-special definite ring.The converse is similar to Theorem 2.8.

Theorem 2.10: Let R be a ring. Then $M_n(R)$ is a S-special definite ring if and only if R is a S-special definite ring.

Proof: Suppose that R is a S-special definite ring, thenby Theorem 2.1 there exists $a\in R$ such that $na\neq 0$, for all $n\in\mathbb{Z}^+$, then $n\begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ for all $n\in\mathbb{Z}^+$, so by Theorem 2.1, $M_n(R)$ is a S-special definite ring.

Conversely suppose that $M_n(R)$ is a S-special definite ring. By Theorem 2.1, there exists $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in M_n(R)$ suchthat $n\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \neq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.

Suppose that every element of (R,+) is of finite order, so for every $a_{ij} \in R$ there exist $m_{ij} \in \mathbb{Z}^+$ such that $m_{ij}\, a_{ij} = 0$. Let $t = m_{11}m_{12}\ldots m_{1n}m_{21}m_{22}\ldots m_{nn}$, so $t\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$, whichis a contradiction, so there exists $a \in R$ such that $n.a \neq 0$, for all $n \in \mathbb{Z}^+$, then R is a S-special definite ring. It is clear that if R has a subring H which is a S-special definite ring, then R is also S-special definite ring but the converse is not true in general as $(\mathbb{Z}\times Z_p,+,.)$ is a S-special definite ring, since it contains the semiring $((\mathbb{Z}^+\cup\{0\})\times Z_p,+,.)$, but the subring $(\{0\}\times Z_p,+,.)$ of $(\mathbb{Z}\times Z_p,+,.)$ is notaS-special definite ring of R. Recall that if R be a S-special definite ring such that every non trivial subring of R is a S-special definite subring, then R is called S - strong special definite ring [3, p.66].

Proposition 2.11: Let R be a S-special definite ring which has no zero divisors. Then R is a S - strong special definite ring.

Proof: Let J be any non zero subring of R. Since R is a S-special definite ring, then there exists $a \in R$ such that $n.a \neq 0$, for all $n \in \mathbb{Z}^+$. If x is a non zero element of J, then $n.x \neq 0$, for all $n \in \mathbb{Z}^+$, since if n.x=0, for some $n \in \mathbb{Z}^+$, then (n.x) a=0, then x. na=0. But $x \neq 0$ and R has no zero divisor, then na=0, which is a contradiction with $na \neq 0$, for all $n \in \mathbb{Z}^+$. Then $x \in J$ and $n.(x) \neq 0$, for all $n \in \mathbb{Z}^+$, then by Theorem 2.1, J is a S-special definitesubring. Then every non trivial subring of R is a S-special definite subring. Then R is a S -strong special definite ring. The converse of Proposition 2.11, is not true in general as $\mathbb{Z} \times \mathbb{Z}$ is a ring which contains zero divisors, but every non zero subring of $\mathbb{Z} \times \mathbb{Z}$ is a S-special definite subring, that is $\mathbb{Z} \times \mathbb{Z}$ is a S - strong special definite ring.

In the following theorem a necessary and sufficient conditionisgiven under which a S-special definite ring is a S-strong special definite ring.

Theorem 2.12: Let R be a S-special definite ring, Then (R,+) is a torsion free group if and only if R is a S- strong special definite ring.

Proof: Suppose that every non trivial subring of R is a S-special definite subring.Let a be a non zero element in R. If $aR \neq \{0\}$, then by assumption aR is a S-special definite subring of R, by Theorem 2.1, for some $b \in R$, ab is an element of infinite order with respect to addition. This implies that a is an element of infinite order with addition, sinceif ma=0, for some $m \in \mathbb{Z}^+$, then m(ab)=(ma)b=0b=0, which is acontradiction. If aR= {0}, then H= {ma; $m \in \mathbb{Z}$) is a S-special definite ring, so byTheorem 2.1, for some $k \in \mathbb{Z}^+$, ka is an element of infiniteorder with respect to addition, consequently a is an element ofinfinite order with respect to addition since if ma=0, for some $m \in \mathbb{Z}^+$, then m(ka)=k(ma) =k0=0, which is acontradiction with ka is anelement of infinite order with addition. Conversely suppose that (R,+) is a torsion free group. Then everynon trivial subring containsan element of infinite order with respect to addition. By Theorem 2.1, every non trivial subring is a S-special definite subring. So R is a S- strong special definite ring.

The following example illustrates Theorem 2.12,

Examples 2.13:

1. $\mathbb{Z} \times \mathbb{Z}$ is a S-special definite ring and $(\mathbb{Z} \times \mathbb{Z}, +)$ is a torsion free group, then by Theorem 2.12, $\mathbb{Z}\times \mathbb{Z}$ is a S- strong special definite ring.
2. $(\mathbb{Z}\times Z_p,+,.)$ is a S-special definite ring and $(\mathbb{Z} \times Z_p, +)$ is not torsion free group, then by Theorem 2.12, $(\mathbb{Z}\times Z_p,+,.)$ is not a S - strong special definite ring.

We would like to mention that the property of being an S - ring and an S-special definite ring, are independent asit is shown in the following example.

Examples2.14:

(1) The infinite direct sum $\oplus Zp$ of the rings Zp, p runs over all prime numbers, is a S-ring but is not a S-special definite ring.

(2) $(\mathbb{Z},+,.)$ is a S-special definite ring but is not a S-ring.

Theorem 2.15: Let R be just a non zero subring of a field F, Then R is a S-special definite ring if and only if F is a field of characteristic zero.

Proof: Suppose that F is a field of characteristic zero and let $0\neq x \in R$. Then $n.x \neq 0$, for all $n\in \mathbb{Z}^+$, since if n.x=0, for some $n\in \mathbb{Z}^+$, then (n.x) a=0, for all $a \in F$ then x. na=0, but $x\neq 0$ and F has no zero divisor, thenn.a=0, for all $a \in F$ which is a contradiction with F is a field of characteristic zero.Thus $x\in R$ and $n.(x)\neq 0$, for all $n\in \mathbb{Z}^+$, thenR isa S-special definitering.

Conversely suppose that R is a S-special definite ring. ThenbyTheorem 2.1, there exists $a\in R\subset F$ such that $na\neq 0$, for all $n\in \mathbb{Z}^+$. Hence F is a field of characteristic zero.

The following example illustrates Theorem 2.15,

Examples 2.16:
1-**(**$\mathbb{Z}$,+,.)is just subring of the field ($\mathbb{Q}$,+,.) whose characteristicis zero, which is S-specialdefinite ring.
2- ($Z_P[x]$, +, .) is just subring of the field $Z_P(x)$ whose characteristic is p, which is not S-special definite ring.

3.S - SPECIAL DEFINITEFIELDS

In this section we study S -special definite fields. We show that a finite field can not be S - Special definite field. We give many characterizations of S - special definite fields. It is shown that every field of characteristic zero is a S-special definite field. Moreoverwe study S - special definite substructures such as S - specialdefinite subfields and S-special definite prime fields and westudy also S - definitespecial fields. It is shown that a field F is a S- definite special fieldif and only if F is of characteristic zero.

Proposition 3.1: A finite field can not be S-special definite field.

Proof: Let F be a finite field and R beasubring of F. Then R-{0}is closed under multiplication. Then (R-{0},.) is a finite semigroup, whichsatisfies cancelation laws. Hence by Theorem 1.1, (R-{0}, .) is a group, thus(R, +, .) is a field, which means that F is not a S-special definite field.

Theorem 3.2: Every field of characteristic zero is a S-special definite field.

Proof: Let F be a field of characteristic zero. Then F contains a subring isomorphic to $\mathbb{Z}$.Hence F is a S-special definitefield.

Now we give a necessary and sufficient condition under whicha field of positive characteristic is S-special definite field.

Theorem 3.3: Let F be a field of characteristic p. Then F is a S-special definite fieldif and only if F is not an algebraic extension of Z_p.

Proof: Suppose that F is not an algebraic extension over Z_p. Then there exists $x\in F$ such that x is transcendental over Z_p. Let $R=\{a_0+ a_1x+\ldots+ a_k x^k ; a_i\in Z_p \text{ and } k\in\mathbb{Z}^+\}$. Then R is a ring. $1.x\in R$ which has no inverse in R, since if 1.x has an inversein R, then there exists $a_0+ a_1x+a_2x^2+\ldots+ a_nx^n\in R$ such that$(1.x)(a_0+ a_1x+\ldots+ a_n x^n)=1$. Then we get $-1+ a_0x+ \ldots+ a_n x^{n+1}=0$, which is a contradiction with x is transcendental over Z_p. HenceR isjust a ring and F is S-special definite field.

Conversely suppose that F is S-special definite field which is analgebraic extension over Z_p. If R is any subring of F and a is a non zero element of R, thena is algebraic over Z_p, then $Z_p(a)$ is a finite field .Suppose $Z_p(a)$ contains n elements, then $(Z_p(a)-\{0\}, .)$ is a cyclic group oforder n-1, then $a^{n-1}=1$, then $a^{-1}= a^{n-2}\in R$, then every non zero element of R has inverse in R. Therefore R is a field, then every subring of F is a subfield, so F cannot be S-special definite field, which is a contradiction with assumption F is S-special definite field. ThenF is not analgebraicextension over Z_p.

Examples3.4:
1. $Z_p(x)$ is a field of characteristic p which is a S-special definite field sinceit contains $Z_p[x]$, which is just a ring.
2. The algebraic closure of Z_p is an algebraic extension of Z_p, then it is not a S-special definite field.
3. ($\mathbb{R}$,+,.) is S-special definite field, since it contains ($\mathbb{Z}$,+,.).
4. No finite field is a S-special definite field.

The following theorem gives another characterization of S-special definite fields.

Theorem 3.5: Let F be a field of characteristic p. Then F is a S-special definite fieldif and only if (F - {0},.) is not a torsion group.

Proof: Suppose that F is a S-special definite field. Then F has a subringR which is not a subfield. So (R-{0},.) is just a semigroup, then there exists an element a in R-{0} such that a has no inversein R-{0}, if a is of finite order with respect to multiplication, thus thereexists $n\in \mathbb{Z}^+$ such that $a^n = 1$, so a $a^{n-1}= 1$, thus $a^{-1}= a^{n-1}\in$ R-{0},which is a contradiction with a has no inversein R-{0}.This means that (R-{0}, .) contains an element of infinite order. Hence (F -{0},.) is not a torsion group.

Conversely suppose that (F -{0}, .) is not a torsion group, then there exist $a\in F$ such that a is an element of infinite order with respect to multiplication. We claim that a is transcendental over Z_p . If a is algebraic over Z_p, then $Z_p(a)$ is a finite fieldof order n. Then ($Z_p(a)$-{0}, .) is a group of order n-1, then $1=a^{n-1}$ which is a contradiction. Then F is notalgebraic extension over Z_p , then F is S-special definitefieldby Theorem 3.3.

Theorem 3.6: If F is a field and R is just a subring of F, then R is an infinite set containing an element of infinite order with respect to multiplication.

Proof: Let F be a field and R be a subring of F which is not a field. If R is a finite set, then R is a finite ring which satisfies cancelation laws. Then by Theorem 1.2, R is a field, which is a contradiction with assumption R is not a field.So R isan infinite set.

Now suppose that every element of R is of finite order with respect to multiplication. Since R is just a ring, then there exists an element $a\neq 0$ in R such that a has no inverse in R, but a is of finite orderwith respect to multiplication, hence there exists $n\in \mathbb{Z}^+$ such that $a^n = 1$, so a $a^{n-1}= 1$, thus $a^{-1}= a^{n-1}\in R$, which is a contradiction. This means that R contains an element of infinite order with respect to multiplication.

Proposition 3.7: Every field can be imbedded in a S-special definite field.

Proof: Let F be a field. Then $F(x) =\{f(x)/g(x);\ f(x), g(x)\in F[x] \text{ and } g(x)\neq 0\}$ is a S-special definite field since it contains the ring F[x]. So, F is imbedded in F(x) , which is a S-specialdefinite field.

Theorem 3.8: Let F be a field. If F is a S-special definite field, then F containsan infinite countable numberof subrings which are not field.

Proof: Let F be a S-special definite field. Then there exists $R \subset F$ such that R is just a ring. Hence there exists $x \in R$ such that $xR \subset R$, since (if xR = R for every non zero element $x\in R$. Then by Theorem 1.3, R is division ring , but R is a commutative ring, so R is a field which is acontradiction with R is just aring). If xR contains the identity 1 (identity of a ring equal the identity of extension field). i.e. $1\in xR$, then there exists $x_l \in R$ such that $xx_l= 1$, so $x^{-1} = x_l\in R$, thus xR =R, since (If $y\in R$, then $y = x (x^{-1}y)\in x R$, thus $R \subseteq xR$ but $xR \subseteq R$, thus x R =R) which is a contradiction with $x R\subset R$, then xR does not contain the identity element. Hence x R is just a ring, which is an infinite set (If x R is afinite, therefore xR is a finite ringand has no zero divisors, then by Theorem 1.2, x R is a field).Then for every justa ring Rthereexists $x \in R$ such that $R_1= x R$ is justa ring which is an infinite set and $R_1\subset R$. By the same mannerone can show the existence of asubring $R_2\subset R_1$which is not a field, then F contains an infinitecountable number of subringswhich arenot field.

Theorem 3.9: Let F be a S-special definite field. Then every subfield of F is a S-special definite subfield if and only if F is a field of characteristic zero.

Proof: Suppose that F is a field of characteristic zero and K is a subfield of F. Then K is a field of characteristic zero and by Theorem 3, 2 K is a S-special definite subfield. Therefore everysubfield of F is a S-special definite subfield.

Conversely, suppose that every subfield of F is S-special definite subfield and F is a field of characteristic p, then F containsa subfield (Zp, +, .) but (Zp, +, .) is not S-special definite field whichis a contradiction with assumption that every subfield of F is a S-specialdefinite subfield, then F is a field of characteristic zero. It is clear that the only S-special definite prime field ofcharacteristic zero is the field of rational numbers but it has no S-special definiteprime field of prime characteristic as it is shown in thefollowing theorem.

Theorem 3.10: There is no S-special definite prime field of characteristic p.

Proof: Let F be a S-special definite field of characteristic p. ByTheorem 3.3, F is notan algebraic extension over Z_p that is there exists $x\in F$ such thatxis transcendental over Z_p, then $x\notin \mathbf{Z_p(x^2)}$**,** since (if $x\in Z_p(x^2)$, then $x= (a_0+ a_2x^2+\ldots+ a_{2n} x^{2n}) /(b_0+ b_2x^2+\ldots+ b_{2n} x^{2r})$ where $b_{2i}\neq 0$ for some i, then$(b_0x+ b_2x^3+\ldots+ b_{2n} x^{2n+1})- (a_0+ a_2x^2+\ldots+ a_{2r} x^{2r})=0$, hencex is algebraic over Z_p which is a contradiction), then $Z_p(x^2) \subset F$. Since x is transcendental over Z_p, then $x^2\in Z_p(x^2)$ is a transcendental over Z_p, thus by Theorem 3.3, $Z_p(x^2)\subset F$ is a S-special definite subfield of F, hence F can not be S-special definite prime field. Then there is noS-special definite prime field of characteristic p.

Theorem 3.11: A field of characteristic zero is a S-definite special field.

Proof : Let F be a field of characteristic zero. Let S={ n.1; n $\in \mathbb{Z}^+$}U{0}. Clearly S is a commutative semiring. For n.1, m 1∈ S, if n 1+m 1=0 where n,m $\in \mathbb{Z}^+$, then (n +m) 1=0, implies (n +m) =0 since F is a field of characteristic zero, so n=m=0, if n 1m 1=0 where n,m $\in \mathbb{Z}^+$, then (n m) 1=0, then (n m) =0, sinceF is a field of characteristic zero, so n=m=0, thus S is a semifield, consequently F is a S-definite special field.

Theorem 3.12: A field of characteristic p is not a S-definite special field.

Proof : Suppose F is a S-definite special field of characteristic p and let S ⊂F be a semifield of F. Then for every element 0≠ a∈ S, we have p a=0, so a+ (p-1) a=0, hence a=0 and (p-1) a=0 which is a contradiction with a ≠0, then F is not a S-definite special field. From Theorem 3.11, and Theorem 3.12, we deduce a necessary and sufficient condition under which a field is S- definite special field.

Corollary3.13: Let F be a field. Then F is a S- definite special field if and only if F is a field of characteristic zero. From Corollary 3.13, and Theorem 3.2 we deduce that every S-definite special field is a S-special definite field but the converse is not true in general as $Z_p(x)$ is a field of characteristic p which is a S-special definite field,since it contains the ring $Z_p[x]$. But $Z_p(x)$ is not a S-definite special field.

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