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Birkoff Centre of a C-algebra

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ABSTRACT

The concept of the Birkoff centre of a Semi group with 0 and 1 was introduced by U.M. Swamy and G.S.N. Murthy [4] and proved that it is a Boolean algebra. This concept is extended to a C-algebra with T. It is proved that Bir A, the Birkoff centre of a C-algebra A is itself a C-algebra. For any element $a \in \mathcal{B}(A)$ we defined S_a and proved that it is a C-algebra.

Key words: *C-algebra, Centre, Birkoff centre, Boolean algebra.*

AMS Mathematics subject classification (2000): *03G25(03G05, 08G05)*

INTRODUCTION:

In [2] Fernando Guzman and Craig C. Squier introduced the variety of C-algebras as the variety generated by the three element algebra C = {T, F, U} with the operations $\wedge$, $\vee$ and ´ of type (2,2,1), which is the algebraic form of the three valued conditional logic. They proved that C and the two element Boolean algebra B = {T, F} are the only sub directly irreducible C-algebras and that the variety of C-algebras is a minimal cover of the variety of Boolean algebras. Later U.M.Swamy et.al., in [6] defined different partial orders on a C-algebra and studied their properties and gave a number of equivalent conditions in terms of this partial ordering for a C-algebra to become a Boolean algebra and in [5], introduced the concept of the Centre $\mathcal{B}(A) = \{ a \epsilon A \mid a \vee a' = T \}$ of a C-algebra A and proved that $\mathcal{B}(A)$ is a Boolean algebra with induced operations on A. Let us recall that S is a Semi group and there exists 0,1 such that $x\,0 = 0 = 0\,x$ and $1\,x = x$, for all $x \epsilon S$ then S is called a Semi group with 0,1. An element $a \epsilon S$ is called Birkoff central element of S if there exists Semi groups S_1 and S_2 with 0 and 1 and an isomorphism S onto $S_1 \times S_2$ which maps a onto (0,1). The set of all Birkoff central elements of S is called Birkoff centre of S. This concept is extended to a C-algebra with T and proved that the set of all central elements of a C-algebra with T is itself a C-algebra. For any element $a \in \mathcal{B}(A)$ we defined S_a and proved that it is a C-algebra.

1. C-algebra:

In this section we recall the definition of a C-algebra and some results from [2], [5] and [6]. Let us start with the definition of a C-algebra.

Definition 1.1: [2] By a C-algebra we mean an algebra of type (2, 2, 1) with binary operations $\wedge$ and $\vee$ and unary operation $'$ satisfying the following identities.

(1) $x'' = x$
(2) $(x \wedge y)' = x' \vee y'$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
(4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$
(6) $x \vee (x \wedge y) = x$
(7) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$.

Example 1.2: [2] The three element algebra C= {T, F, U} with the operations given by the following tables is a C-algebra.

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∧	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

∨	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

x	x'
T	F
F	T
U	U

Note 1.3: [2] The identities 1.1(1), 1.1(2) imply that the variety of C-algebras satisfies all the dual statements of 1.1(2) to 1.1(7). ∧ and ∨ are not commutative in C.The ordinary distributive law of ∧ over ∨ fails in C. Every Boolean algebra is a C-algebra.

Now we recall some results on C-algebra collected from [2], [5] and [6].

Lemma 1.4: Every C-algebra satisfies the following identities:

(1) $x \wedge x = x$ (2) $x \wedge x' = x' \wedge x$
(3) $x \wedge y \wedge x = x \wedge y$ (4) $x \wedge x' \wedge y = x \wedge x'$
(5) $x \wedge y = (x' \vee y) \wedge x$ (6) $x \wedge y = x \wedge (y \vee x')$
(7) $x \wedge y = x \wedge (x' \vee y)$ (8) $x \wedge y \wedge x' = x \wedge y \wedge y'$
(9) $(x \vee y) \wedge x = x \vee (y \wedge x)$ (10) $x \wedge (x' \vee x) = (x' \vee x) \wedge x = (x \vee x') \wedge x = x.$

Duals of the statements in the above lemma are also true in a C-algebra.

Definition 1.5: [5] Let A be a C-algebra with T (T is the identity element for ∧ in A). Then the Boolean centre of A is defined as the set $\mathcal{B}(A) = \{ a \in A \mid a \vee a' = T \}$. $\mathcal{B}(A)$ is known to be a Boolean algebra under the operations induced by those on A.

Lemma 1.6: [5] Every C -algebra with T satisfies the law $x \wedge F = x \wedge x'$.

Lemma 1.7: [5] If A is a C -algebra with T and $a \in \mathcal{B}(A)$ then $a \wedge a' = F$.

2. Birkoff Centre:

In this section we define Birkoff centre of a C-algebra and we shall prove various properties. First let us start with the following definition of Birkoff central element.

Definition 2.1: Let A be a C-algebra with ∧-identity T. An element $a \in A$ is said to be Birkoff central element of a C-algebra A if there exists C-algebras A_1 and $A_2$with T and an isomorphism $f: A \to A_1 \times A_2$ such that $f(a) = (T_1, F_2)$.

Definition 2.2: The set of all Birkoff central elements of a C-algebra A is called Birkoff centre of A and is denoted by Bir A.

Lemma 2.3: Let A be a C-algebra with T. Then $a \in Bir\ A \Rightarrow a' \in$ Bir A.

Proof: Let $a \in$ Bir A then there exist C-algebras A_1, A_2 and an isomorphism $\alpha: A \to A_1 \times A_2$ such that $\alpha(a) = (T_1, F_2)$.

Now define $g: A \to A_2 \times A_1$ such that $g(x) = (x_2, x_1)$ whenever $\alpha(x) = (x_1, x_2)$.

Let $x, y \in A$ such that $\alpha(x) = (x_1, x_2)$, $\alpha(y) = (y_1, y_2)$. Then $\alpha(x \wedge y) = (x_1 \wedge y_1, x_2 \wedge y_2)$.

Now, $g(x \wedge y) = (x_2 \wedge y_2, x_1 \wedge y_1) = (x_2, x_1) \wedge (y_2, y_1) = g(x) \wedge g(y)$.

Similarly, we can prove $g(x \vee y) = g(x) \vee g(y)$.

To show $g(x') = [g(x)]'$. Let $\alpha(x) = (x_1, x_2)$. Then $g(x) = (x_2, x_1)$.

$\alpha(x) = (x_1, x_2)$

$\Rightarrow\ (\alpha\,(x))' = (x_1, x_2)' = (x_1', x_2')$

$\Rightarrow\ \alpha\,(x') = (x_1', x_2')$ (since α is a homomorphism)

$\Rightarrow g\,(x') = (x_2', x_1') = (x_2,\ x_1)' = (g(x))'$

Therefore g is a homomorphism. Also $\alpha(a') = (\alpha(a))' = (T_1, F_2)' = (F_1, T_2)$.

Then $g(a') = (T_2,\ F_1)$. Thus $a' \epsilon$ Bir A . Clearly g is bijective. Therefore g is an isomorphism.

Lemma 2.4: Let A be a C -algebra and $t \epsilon A$ then $tA = \{\, t \wedge \alpha \mid \alpha \in A\}$ is itself a C - algebra by induced operations $\wedge$ and $\vee$ of A and the unary operation defined by $(t \wedge \alpha)^* = t \wedge \alpha'$.
Proof is a routine verification.

Lemma 2.5: Bir A is a C-algebra.

Proof: Let $a, b \in$ Bir A. Then there exist C-algebras A_1, A_2 and A_3, A_4 with T and isomorphisms $f: A \to A_1 \times A_2$ such that $f\,(a) = (T_1, F_2)$ and $g: A \to A_3 \times A_4$ such that $g(b) = (T_3, F_4)$.

Now we have to prove that $a \wedge b \,\epsilon$ Bir A that is we have to find an isomorphism $h: A \to A_5 \times A_6$ such that $h(a \wedge b) = (T_5,\ F_6)$. Let $g(a) = (t_3,\ t_4)$ where $t_3 \in A_3$ and $t_4 \in A_4$. Now put $A_5 = t_3\ A_3$ where t_3 is meet identity in A_3, $t_3 \wedge t_3'$ is join identity and $t_3\ A_3 = \{\, t_3 \wedge \alpha \mid \alpha \in A\,\}$. By lemma 2.4, $t_3\ A_3$ is a C-algebra with $T = t_3$. Also put $A_6 = t_4\ A_4 \times A_2$, which is also a C-algebra with meet identity $T_6 = (t_4, T_2), F_6 = (t_4 \wedge t_4',\ F_2)$.

For any $x \,\epsilon\, A$, let $f(x) = (s_1,\ s_2)$ and $g(x) = (x_3, x_4)$ where $x_3 \,\epsilon\, A_3, x_4 \,\epsilon\, A_4$ and $s_1 \,\epsilon\, A_1, s_2 \,\epsilon\, A_2$. Now define $h: A \to A_5 \times A_6$ by $h(x) = (t_3 \wedge x_3, (t_4 \wedge x_4,\ s_2))$, for any $x \,\epsilon\, A$.
Then h is well defined.

Let $f(y) = (r_1,\ r_2)$ and $g(y) = (y_3, y_4)$. Then $f(x \wedge y) = (s_1 \wedge r_1,\ s_2 \wedge r_2)$,

$g(x \wedge y) = (x_3 \wedge y_3, x_4 \wedge y_4), f(x') = (s_1',\ s_2')$ and $g(x') = (x_3', x_4')$.

$$\begin{aligned}
h(x \wedge y) &= (t_3 \wedge x_3 \wedge y_3), (t_4 \wedge x_4 \wedge y_4,\ s_2 \wedge r_2)) \\
&= (t_3 \wedge x_3 \wedge t_3 \wedge y_3), (t_4 \wedge x_4 \wedge t_4 \wedge y_4,\ s_2 \wedge r_2)) \quad \text{(by lemma 1.4(3))} \\
&= (t_3 \wedge x_3, (t_4 \wedge x_4,\ s_2)) \wedge (t_3 \wedge y_3, (t_4 \wedge y_4,\ r_2)) \\
&= h(x) \wedge h(y)
\end{aligned}$$

$$\begin{aligned}
\text{Now } h(x') &= (t_3 \wedge x_3', (t_4 \wedge x_4',\ s_2')) \qquad \text{(since } (t_3 \wedge x_3)^* = t_3 \wedge x_3'\text{)} \\
&= (x_3^*, (x_4^*,\ s_2')) \\
&= (h(x))'
\end{aligned}$$

$$\begin{aligned}
h(x \vee y) &= (t_3 \wedge (x_3 \vee y_3), (t_4 \wedge (x_4 \vee y_4),\ s_2 \vee r_2)) \\
&= ((t_3 \wedge x_3) \vee (t_3 \wedge y_3)((t_4 \wedge x_4) \vee (t_4 \wedge y_4),\ s_2 \vee r_2)) \\
&= (t_3 \wedge x_3, (t_4 \wedge x_4,\ s_2)) \vee (t_3 \wedge y_3, (t_4 \wedge x_4,\ r_2)) \\
&= h(x) \vee h(y)
\end{aligned}$$

Therefore h is a homomorphism.

To show h is one-one, first we prove $h(a \wedge b) = (T_5,\ F_6)$.

We have $f(a) = (T_1, F_2),\ g(a) = (t_3,\ t_4),\ g(b) = (T_3, F_4),\ f(b) = (n_1,\ n_2)$.

$$\begin{aligned}
\text{Now } h\,(a \wedge b) &= h(a) \wedge h(b) \qquad \text{(since } h \text{ is a homomorphism)} \\
&= (t_3 \wedge t_3, (t_4 \wedge t_4, F_2)) \wedge (t_3 \wedge T_3, (t_4 \wedge F_4,\ n_2)) = (t_3, (t_4 \wedge F_4, F_2)) \\
&= (t_3,\ t_4 \wedge t_4',\ F_2) \quad \text{(since by Lemma 1.5)} \\
&= (T_5,\ F_6)
\end{aligned}$$

Let $x, y \in A$ such that $h(x) = h(y)$. Then $t_3 \wedge x_3 = t_3 \wedge y_3$, $t_4 \wedge x_4 = t_4 \wedge y_4$ *and* $s_2 = r_2$.

$$\begin{aligned}
\text{Now } g(a) \wedge g(x) &= (t_3, t_4) \wedge (x_3, x_4) \\
&= (t_3 \wedge x_3, t_4 \wedge x_4) \\
&= (t_3 \wedge y_3, t_4 \wedge y_4)) \\
&= g(a) \wedge g(y)
\end{aligned}$$

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Since g is a homomorphism, $g(a \wedge x) = g(a \wedge y)$

$\Rightarrow a \wedge x = a \wedge y$ (since g is one-one)
$\Rightarrow f(a \wedge x) = f(a \wedge y)$ (since f is well defined)
$\Rightarrow f(a) \wedge f(x) = f(a) \wedge f(y)$ (since f is a homomorphism)
$\Rightarrow (T_1, F_2) \wedge (s_1, s_2) = (T_1, F_2) \wedge (r_1, r_2)$
$\Rightarrow (T_1 \wedge s_1, F_2 \wedge s_2) = (T_1 \wedge r_1, F_2 \wedge r_2)$
$\Rightarrow (s_1, F_2) = (r_1, F_2)$
$\Rightarrow s_1 = r_1, s_2 = r_2$
$\Rightarrow (s_1, s_2) = (r_1, r_2)$
$\Rightarrow f(x) = f(y)$
$\Rightarrow x = y$ (since f is one-one)

Therefore h is one-one.

Let $(x, y) \in A_5 \times A_6$. Then $(x, y) = (t_3 \wedge x_3, (t_4 \wedge x_4, s_2))$ for some $x_3 \in A_3, x_4 \in A_4, s_2 \in A_2$.

Since, $t_3 \wedge x_3 \in t_3 A_3 \subseteq A_3, (t_3 \wedge x_3, t_4 \wedge x_4) \in A_3 \times A_4$ and g is onto, there exists $t \in A$ such that $g(t) = (t_3 \wedge x_3, t_4 \wedge x_4)$.

Now $g(a \wedge t) = g(a) \wedge g(t)$
$= (t_3, t_4) \wedge (t_3 \wedge x_3, t_4 \wedge x_4)$
$= (t_3 \wedge t_3 \wedge x_3, t_4 \wedge t_4 \wedge x_4)$
$= (t_3 \wedge x_3, t_4 \wedge x_4)$
$= g(t)$

Therefore $g(a \wedge t) = g(t)$ (1)

$\Rightarrow a \wedge t = t$ (since g is one-one)
$\Rightarrow f(a \wedge t) = f(t)$ (since f is well defined)
$\Rightarrow f(a) \wedge f(t) = f(t)$ (since f is a homomorphism)
$\Rightarrow (T_1, F_2) \wedge (y_1, y_2) = (y_1, y_2)$ (since $t \in A$)
$\Rightarrow f(t) = (y_1, y_2)$
$\Rightarrow (T_1 \wedge y_1, F_2 \wedge y_2) = (y_1, y_2)$
$\Rightarrow (y_1, F_2) = (y_1, y_2)$
$\Rightarrow y_2 = F_2$ (2)

Now, $y_1 \in A_1$ *and* $s_2 \in A_2$ *then* $(y_1, s_2) \in A_1 \times A_2$. Since f is onto there exists $n \in A$ such that $f(n) = (y_1, s_2)$.

Now $f(a \wedge n) = f(a) \wedge f(n)$
$= (T_1, F_2) \wedge (y_1, s_2)$
$= (T_1 \wedge y_1, F_2 \wedge s_2)$
$= (y_1, F_2)$
$= f(t)$ (by (2))

Since f is one-one $a \wedge n = t$ and g is well defined $g(a \wedge n) = g(t)$ (3)

Also $n \in A \Rightarrow g(n) = (z_1, z_2)$
$(t_3 \wedge t_3 \wedge x_3, t_4 \wedge t_4 \wedge x_4) = g(a \wedge t)$
$= g(t)$ (since by (1))
$= g(a \wedge n)$ (since by (3))
$= g(a) \wedge g(n)$ (since g is a homomorphism)
$= (t_3, x_4) \wedge (z_1, z_2)$
$= (t_3 \wedge z_1, t_4 \wedge z_2)$.

Therefore $t_3 \wedge t_3 \wedge x_3 = t_3 \wedge z_1$ and $t_4 \wedge t_4 \wedge x_4 = t_4 \wedge z_2$ (4)

Now, $h(n) = (t_3 \wedge z_1, (t_4 \wedge z_2, s_2))$
$= (t_3 \wedge t_3 \wedge x_3, (t_4 \wedge t_4 \wedge x_4, s_2))$ (since by (4))
$= \left(t_3 \wedge x_3, (t_4 \wedge x_4, s_2)\right)$
$= (x, y)$

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Therefore h is onto. Since $a, b \in$ Bir A imply $a \wedge b \in$ Bir A and by Lemma 2.3, $a \in$ Bir A $\Rightarrow a' \in$ Bir A also $a \vee b$ also belong to Bir A. Therefore Bir A is sub algebra of a C-algebra A and hence Bir A is a C-algebra.

Let us recall the definition of Centre of a C-algebra defined in [5].

Let A be a C-algebra with identity T. Then the Centre of A is defined as the set

$\mathcal{B}(A) = \{ a \epsilon A \mid a \vee a' = T \}$. $\mathcal{B}(A)$ is known to be a Boolean algebra under the operations induced by those on A.

Lemma 2.6: Let $a \in \mathcal{B}(A)$ then $a \wedge x = a \Leftrightarrow a \vee x = x$.

Proof:
$$\begin{aligned} a \vee x &= (a \wedge x) \vee x \\ &= (a \vee x) \wedge (a' \vee x) \\ &= (a \wedge a') \vee x \\ &= F \vee x \qquad \text{(by Lemma 1.6)} \\ &= x \end{aligned}$$

Conversely, $a \wedge x = a \wedge (a \vee x) = a$.

Now, for any C-algebra A and $a \in \mathcal{B}(A)$ we define S_a and prove that it is a C-algebra.

Lemma 2.7: Let A be a C-algebra A and $a \in \mathcal{B}(A)$. If $S_a = \{x \in A \mid a \wedge x = a\}$ then S_a is a C-algebra.

Proof: Let $x, y, z \in S_a$. Then $a \wedge x = a,\ a \wedge y = a \; and \; a \wedge z = a$

Since $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = a \vee a = a, x \vee y \in S_a$

Also, since $a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) = a \wedge a = a, x \wedge y \in S_a$

Define $x^* = a \vee x'$. Since $a \wedge (a \vee x') = a',\ a \vee x' \in S_a$

$$\begin{aligned} x^{**} &= (a \vee x')^* \\ &= a \vee (a \vee x')' \qquad \text{(by Definition 1.1(2))} \\ &= a \vee (a' \wedge x) \\ &= a \vee x \qquad \text{(by dual of 1.4(7))} \\ &= (a \wedge x) \vee x \\ &= (a \vee x) \wedge (a' \vee x \vee x) \\ &= (a \wedge a') \vee x \\ &= x \end{aligned}$$

$(x \wedge y)^* = a \vee (x \wedge y)' = a \vee (x' \vee y') = a' \vee x' \vee a \vee y' = x^* \vee y^*$.

$a \wedge [(x \vee y) \wedge z] = a \Rightarrow a \vee [(x \vee y) \wedge z] = (x \vee y) \wedge z$ (by Lemma 2.6)

Now,
$$\begin{aligned} (x \vee y) \wedge z &= a \vee [(x \vee y) \wedge z] \\ &= a \vee [(x \wedge z) \vee (x' \wedge y \wedge z)] \\ &= a \vee (x \wedge z) \vee (x' \wedge y \wedge z) \\ &= (a \vee x) \wedge (a \vee z) \vee [(a \vee x') \wedge (a \vee (y \wedge z))] \\ &= (x \wedge z) \vee [(a \vee x') \wedge (a \wedge y) \wedge (a \vee z)] \\ &= (x \wedge z) \vee [x' \wedge y \wedge z]. \end{aligned}$$

Therefore S_a is a C-algebra.

Lemma 2.8: Let A be a C-algebra and $a \in \mathcal{B}(A)$. Then $f_a: A \to S_a$ is an antihomorphism.

Proof: Define $f_a: A \to S_a \; by \; f_a(x) = a \vee x'$

$$f_a(x \wedge y) = a \vee (x \wedge y)' = a \vee (x' \vee y') = (a \vee x') \vee (a \vee y') = f_a(x) \vee f_a(y)$$

$$f_a(x \vee y) = a \vee (x' \wedge y') = (a \vee x') \wedge (a \vee y') = f_a(x) \wedge f_a(y)$$

$[f_a(x)]^* = (a \vee x')^* = a \vee (a \vee x')' = a \vee (a' \wedge x) = a \vee x = a \vee (x')' = f_a(x')$.
Therefore $f_a: A \to S_a$ is an antihomorphism.

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