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AN IDENTIFICATION OF $S_6(2)$

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ABSTRACT

Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $nse(G) = \{ s_k \mid k \in \omega(G) \}$. In Khatami et al and Liu, the authors proved that $L_3(2)$ and $L_3(4)$ are unique determined by $nse(G)$. In this note, we prove that if G is a group such that $nse(G) = nse(S_6(2))$, where $S_6(2)$ is the projective sympletic group of degree 6 over fields of order 2, then $G \cong S_6(2)$.

Keywords: *Element order, Unitary group, Thompson' problem, Number of elements of the same order.*

Mathematics Subject Classification (2000): *20D05, 20D06, 20D20.*

1. INTRODUCTION

We introduce some which may be unfamiliar to the reader. Let $\omega(G)$ denote the set of element orders of G. Let $m_i(G) := |\{ g \in G \mid \text{the order of } g \text{ is } i \}|$ (m_i for short), be the number of elements of order i, and let $nse(G) := \{ m_i(G) \mid i \in \omega(G) \}$, the set of sizes of elements with the same order. $n_p(G)$ denotes the number of Sylow p-subgroup of G, namely, $n_p(G) = |Syl_p(G)|$. $\pi(G)$ denotes the set of all prime divisors of $|G|$. We use $a \mid b$ to mean that a divides b; if p is a prime, then $p^n \| b$ means $p^n \mid b$ but $p^{n+1} \nmid b$. $\mathbb{N} = \{1, 2, 3, 4\}$ denotes the set of positive integers. $\pi(G)$ denotes the set of prime divisors of $|G|$ and $|\pi(G)|$ the number of the element of the set $\pi(G)$. $nse(G)$ denotes the number of elements of a given order of G.

For the set $nse(G)$, the most important problem is related to the Thompson's problem. In 1987, J.G. Thompson put forward the following problem.

Thompson's problem: For each finite group G and each integer $d \geq 1$, let $G(d) = \{ x \in G \mid x^d = 1 \}$. Define G_1 and G_2 are of the same order type if and only if $|G_1(d)| = |G_2(d)|$, d =1, 2, 3,…Suppose G_1 and G_2 are of the same order type. If G_1 is solvable, is G_2 necessarily solvable?

It was proved that: Let G be a group and M some simple k_i-group, i=3, 4, then $G \cong M$ if and only if $|G| = |M|$, and $nse(G) = nse(M)$ (see [7,8]) Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson Problem, in other words, it remains only $nse(G)$, whether can it characterize finite simple groups? Up to now, some groups especial for $PSL(2, q)$, can be characterized by only the set $nse(G)$ (see [3],[5] and [10]). Recent, $L_3(4)$ is characterization by $nse(L_3(4))$ (see [6]). In this paper, it is shown that projective symplectic group $S_6(2)$ can be characterized by $nse(S_6(2))$, that is:

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Main Theorem: Let G be a group. If $nse(G) = nse(S_6(2))$. Then $G \cong nse(S_6(2))$.

2. PRELIMINARY

Lemma 2.1: [1] Let G be a finite group and m be a positive integer dividing $|G|$. If , $L_m(G) = \{g \in G / g^m = 1\}$. then $m \mid |L_m(G)|$

Lemma 2.2: [2] Theorem 9.3.1] Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1}, \cdots p_r^{\alpha_r}$, $(m,n) = 1$ Let $\pi = \{p_1, \cdots p_r\}$ and h_m be the number of Hall π -subgroups of G. The $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots s\}$:

(1) $q_i^{\beta_i} \equiv 1$ (mod p_j) for some p_j .

(2) The order of some chief factor of G is divided by $q_i^{\beta_i}$

Lemma 2.3: [4] Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p -subgroup of G and $n = p^s m$ with $(p,m) = 1$ If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s .

Lemma 2.4: [10] Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \le s(s^2 - 1)$.

The following Lemma is used without referee.

Lemma 2.5: [9] Let G be a finite group, $P \in Syl_p(G)$, where $p \in \pi(G)$. Suppose that G has a normal series $K \lhd L \lhd G$ and the following statements hold:

(1) $N_{G/K}(PK/K) = N_G(P)K/K$.

(2) If $P \le L$, then $|G : N_G(P)| = |L : N_L(P)|$, namely, $n_p(G) = n_p(L)$.

(3) If $P \le L$,then $|L/K : N_{L/K}(PK/K)| t = |G : N_G(P)| = |L : N_L(P)|$, namely, $n_p(L/K) = t = n_p(G) = n_p(L)$. In particular, $|N_K(P)| t = |K|$.

3. THE PROOF OF MAIN THEOREM

Let G be a group such that $nse(G) = nse(J_3)$, and s_n be the number of elements of order n . By Lemma 2.4 we have G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n . Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$\begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d|m} s_d \end{cases} \tag{1}$$

We rewrite the Main Theorem here.

MAIN THEOREM

Let G be a group. If $nse(G) = nse(S_6(2))$ ={ 1, 5103, 16352, 48384, 75600, 96768, 145152, 161280, 181440, 207360, 241920, 272160}, then Then $G \cong S_6(2)$.

Proof: We prove the theorem by first proving that $\pi(G) \subseteq \{2, 3, 5, 7\}$, second showing that $|G|=|S_6(2)|$, and so $G \cong S_6(2)$.

By (1), $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 127, 96769, 161281, 241921\}$. if $k>2$, then $\phi(k)$ is even, then s_2 =5103, $2 \in \pi(G)$.In the following, we prove that 19 $\notin \pi(G)$. If $11 \in \pi(G)$ then by (1), s_{11}=207360. If $2 \cdot 11 \in \omega(G)$, then $s_{22} \notin nse(G)$.Therefore $38 \notin \omega(G)$. Now we consider Sylow 19-subgroup P_{11} acts fixed point freely on the set of elements of order 2, then $|P_{11}| \mid s_2$ (=5103), a contradiction. So $11 \notin \pi(G)$. Similarly, we can prove that 13, 17, 127, 96769, 161281, and $241921 \notin \pi(G)$.

If $2^a \in \omega(G)$ then by (1), $\phi(2^a) \mid s_{2^a}$ and so $0 \le a \le 10$.

If $3^a \in \omega(G)$ then $1 \le a \le 6$.

If $5^a \in \omega(G)$, then $1 \le a \le 3$. If $5^3 \in \omega(G)$ then by (1) $s_{5^3} \notin nse(G)$ since s_{5^2} =181440.

Therefore $1 \le a \le 2$.

If $19^a \in \omega(G)$ then $a = 1$.

To remove the prime 19, the fact that the prime 3 belongs to $\pi(G)$ is proved.

Assume that 3 does not belong to $\pi(G)$.

If 5, 7 and 19 do not belong to $\pi(G)$, then G is a 2-group. Hence 1451520 + 16352 k_1 + 48384 k_2 + 75600 k_3 + 96768 k_4 + 145152 k_5 + 161280 k_6 + 181440 k_7 + 207360 k_8 + 241920 k_9 + 272160 k_{10} = 2^l where k_i, i =1, 2, …10, and I are non-negative integers. But $|\omega(G)|$=11, so the equation has no solution in N.

Let $5 \in \pi(G)$, then $\exp(P_5)$ =5 or 25.

Let $\exp(P_5)$ =5, then by Lemma 2.1, $|P_5| \mid 1+s_5$ and so $|P_5|$=5. It follows that $n_5 = s_5/\phi(5)$ =48384/4= $2^6 \cdot 3^3 \cdot 7$ and so $3 \in \pi(G)$, a contradiction.

Let $\exp(P_5)$ =25, then $|P_5| \mid 1+s_5+s_{25}$ and so $|P_5|$ =25. Thus $n_5 = s_{25}/\phi(5)$ =181440/20 = $2^4 \cdot 3^4 \cdot 7$ and 3 $\in \pi(G)$, a contradiction.

Let $7 \in \pi(G)$, then $\exp(P_7)$ =7 or 49.

Let $\exp(P_7)$ =7, then $|P_7| \mid 1+s_7$ and so $|P_7|$=7. Thus $n_7 = s_7/\phi(7)$ =207360/6 $2^8 \cdot 3^3 \cdot 5$ and so $3 \in \pi(G)$, a contradiction.

Let $\exp(P_7)$ = 49, then $|P_7| \mid 1+s_7+s_{49}$ and so$|P_7|$=49. Thus $n_7 = s_{7^2}/\phi(7^2)$ =241920/42= $2^7 \cdot 3^2 \cdot 5$ and so 3 $\in \pi(G)$, a contradiction.

Let $19 \in \pi(G)$,then as $\exp(P_{19}) = 19$, $n_{19} = s_{19}$ =75600/18. Therefore $3 \in \pi(G)$, a contradiction.

Therefore $3 \in \pi(G)$. In particularly, if 5 $\in \pi(G)$, then 2, 3, 7 $\in \pi(G)$, if $7 \in \pi(G)$, then 2, 3, $5 \in \pi(G)$, a contradiction.

If $57 \in \pi(G)$, then by lemma 2.1 57 ∣ 1+ $s_3 + s_{19} + s_{57}$ .and so $s_{57} \notin nse(G)$ $. It follows that the Sylow 19-subgroups of G acts fixed freely on the set of order 3, and so $|P_{13}|$ ∣ s_3, a contradiction. Thus $19 \notin \pi(G)$.

Therefore the following two cases are considered :{ 2, 3} and {2, 3, 5, 7}.

Case A: $\pi(G)$ = {2, 3}.

By Lemma 2.1,| P_2 | ∣ 1+ $s_2 + \cdots s_{10}$ and so | P_2 | ∣ 2^{10}.

It is easy to see that $\exp(P_3)$ = 3, 9, 27, 81, 243, 729.

Let $\exp(P_3)$ =3, then $|P_3|$ ∣ 1+ s_3 and so $|P_3|$ ∣ 9.If | P_3 |=3, then $n_3 = s_3/\phi(3)$ =16352/2 and so $7 \in \pi(G)$, a contradiction. If If $|P_3|$=9, then 1451520 + 16352 k_1 + 48384 k_2 + 75600 k_3 + 96768 k_4 + 145152 k_5 + 161280 k_6 + 181440 k_7 + 207360 k_8 + 241920 k_9 + 272160 $k_{10} = 2^l \cdot 9$ where k_i, i =1, 2, …10, and Ⅰ are non-negative integers, and $0 \le \sum_{i=1}^{10} k_i \le 25$. Since 1451520 ≤ $|G| = 2^l \cdot 9$ |≤ 1451520+25.272160, then Ⅰ =14, 15, 16, a contradiction as Ⅰ is at most 10.

Let $\exp(P_3)$ =9, then $|P_3|$ ∣ 1+ $s_3 + s_9$ and so $|P_3|$ ∣ 27.If | P_3 |=9, then $n_3 = s_3/\phi(3^2)$ and so 5 or 7 belongs to $\pi(G)$ a contradiction. If $|P_3|$=27,then 1451520 + 16352 k_1 + 48384 k_2 + 75600 k_3 + 96768 k_4 + 145152 k_5 + 161280 k_6 + 181440 k_7 + 207360 k_8 + 241920 k_9 + 272160 $k_{10} = 2^l \cdot 27$ where k_i, i =1, 2, …10, and Ⅰ are non-negative integers, and $0 \le \sum_{i=1}^{10} k_i \le 35$. Since 1451520 ≤ $|G| = 2^l \cdot 27$ ≤ 1451520+35.272160, then Ⅰ =13, 14, 15, a contradiction as Ⅰ is at most 10.

Let $\exp(P_3)$ =27, then $|P_3|$ ∣ 1+ s_3 + s_9 + s_{27} and so $|P_3|$ ∣ 3^6 .If | P_3 |=27, then 5 or 7 belongs to $\pi(G)$, a contradiction. If $|P_3|$ = 3^4 , then 1451520+16352 k_1 +48384 k_2 +75600 k_3 +96768 k_4 +145152 k_5 +161280 k_6 +181440 k_7 +207360 k_8 +241920 k_9 +272160 $k_{10} = 2^l \cdot 81$ where k_i, i =1, 2, …10, and Ⅰ are non-negative integers, and $0 \le \sum_{i=1}^{10} k_i \le 45$.

Since 1451520 ≤ $|G| = 2^l \cdot 81$ ≤ 1451520+45.272160, then Ⅰ =15, 16, 17, a contradiction as Ⅰ is at most 10. If $|P_3|$ =243, then Ⅰ = 13, 14, 15, a contradiction. If $|P_3|$=729, then Ⅰ =11, 12, 13, 14, a contradiction.

Let $\exp(P_3)$ =81, then $|P_3|$ ∣ 1+ $s_3 + s_9$ + $s_{27} + s_{81}$ and so $|P_3|$ ∣ 3^7 .If $|P_3|$=81, then 5 or 7 belongs to $\pi(G)$, a contradiction. If $|P_3|$ = 3^5 ,then 1451520+16352 k_1 +48384 k_2 +75600 k_3 +96768 k_4 +145152 k_5 +161280 k_6 +181440 k_7 +207360 k_8 +241920 k_9 +272160 k_{10} = $2^l \cdot 243$ where k_i, i =1, 2, …10, and Ⅰ are non-negative integers, and. $0 \le \sum_{i=1}^{10} k_i \le 55$. Since 1451520 ≤ $|G|$ = $2^l \cdot 243$ ≤ 1451520+55.272160, then Ⅰ =13, 14, 15, a contradiction as Ⅰ is at most 10. If $|P_3|$=729, then Ⅰ =11, 12, 13, 14, a contradiction. If | P_3 |=2187, then Ⅰ =10, 11, 12. In this case we contradiction Ⅰ =10 only. So $|G| = 2^{10} \cdot 2187$. The number of Sylow 3-subgroup of G is 1, 4, 16, 64, 256, 1024 and so the number of order 3 is 2, 32, 128, 512, 2048, but none of which is in $nse(G)$, so we rule out this case.

Let $\exp(P_3)$ =243, then by (1), s_{243} =181440. By Lemma 2.1, $|P_3| \mid 1+ s_3 + s_9 + s_{27} + s_{81} + s_{243}$ and so $|P_3| \mid 3^{12}$.

Let $\exp(P_3)$ =243, then $n_3 = s_{243}/\phi(243) = 2^5 \cdot 5 \cdot 7$ and so 5 or 7 belongs to $\pi(G)$, a contradiction. Similarly we can rule out the order cases $|P_3| = 3^7, 3^8, 3^9, 3^{10}, 3^{11}, 3^{12}$ as the methods of " $\exp(P_3)$ =81, $|P_3|$=2187 and l =10 ''.

Let $\exp(P_3)$ =729, then $|P_3| \mid 1+ s_3 + s_9 + s_{27} + s_{81} + s_{243}$ and so $|P_3| \mid 3^7$ .If $|P_3|$ =729, then $n_3 = s_{729}/\phi(729)$ =272160/486=16.5.7, and so 5 or 7 belongs to $\pi(G)$, a contradiction. If $|P_3| = 3^7$, then by Lemma 2.3, $s_{729} = 729t$ for some integer t, but the equation $s_{729} = 729t$ has no solution in $\mathbb{N}$.

Case B: $\pi(G)$ = {2, 3, 5, 7}.
We show that $2.7 \notin \omega(G)$.

If $2.7 \in \omega(G)$ set P and Q are Sylow 2-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G. Therefore we have $s_{2.7} = \phi(14).n_7 \cdot k$, where k is the number of cyclic subgroups of order 19 in $C_G(P_7)$. As $n_7 = s_7/\phi(7)$ =207360/6, 207360 $\mid s_{14}$ and so $s_{14} = s_7$.

But by Lemma 2.1, 14 $\mid 1+ s_2 + s_7 + s_{14}$, a contradiction. We conclude that $2.7 \notin \omega(G)$ .It follows that the group P_2 acts fixed point freely on the set of elements of order 7 and so $|P_2| \mid s_7$. so we have $|P_2| \mid 2^9$.

Similarly we have that: $3.7 \notin \omega(G)$ and $|P_3| \mid 3^4$; $5.7 \notin \omega(G)$, $|P_5|$=5 and $|P_7|$=7.

Since $1451520 \le |G| = 2^l \cdot 3^m \cdot 5 \cdot 7$, then I=9, m=4 and so $|G| = |S_6(2)|$. By assumption, $nse(G) = nse(S_6(2))$, and so by [7], $G \cong S_6(2)$.

This completes the proof.

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