



SOME GENERATING FUNCTIONS INVOLVING OF TWO VARIABLE LAGUERRE POLYNOMIALS

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ABSTRACT

Laguerre polynomials have special importance in engineering, science and good model for many systems in various fields. By means of Weisner's group theoretic method some new generating functions of two variable and one parameter Laguerre polynomials $L_n^\alpha(x, y)$ are obtained from which several generating functions can be easily derived.

In this paper we change the order of $A_{i1}, A_{i2}, A_{i3}, i = 1, 2, \dots$ in the operator $e^{a_{23}A_{23}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} e^{a_{11}A_{11}}$ in $L_n^\alpha(x, y)Z^n$, we get other type of generating functions involving Laguerre polynomials.

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1. INTRODUCTION

Group theoretic method was proposed by Louis Weisner in 1955 and he employed this method to find generating relations for a large class of special functions.

Weisner discussed the group-theoretic significance of generating functions for hypergeometric, Hermite and Bessel functions [4,5 and 6] respectively. Miller, McBride, Srivastava and Monocha [3,7 and 8] respectively reported group theoretic method for obtaining generating relations in their books.

Two variables and one parameter Laguerre polynomials $L_n^\alpha(x, y)$ have been defined in [9] and specified by the series

$$L_n^\alpha(x, y) = \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_k x^k y^{n-k}}{k!(n-k)!(1+\alpha)_k},$$

where $\Gamma(\alpha) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ is the pochhammer symbol and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, (\text{Re}(\alpha) > 0)$, and the generating function for $L_n^\alpha(x, y)$ is given by

$$\sum_{n=0}^\infty L_n^\alpha(x, y) t^n = \frac{1}{(1-yt)^{1+\alpha}} \exp\left(\frac{-xt}{1-yt}\right), \quad |yt| < 1$$

where α is a non-negative integer.

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The differential equation satisfied by one parameter and two variables Laguerre polynomials $L_n^\alpha(x, y)$ is:

$$\left[x \frac{d^2}{dx^2} + \left(1 + \alpha - \frac{x}{y} \right) \frac{dy}{dx} \right] L_n^\alpha(x, y) = 0$$

In a recent paper, we have derived the following main generating functions involving Laguerre polynomials:

$$\begin{aligned} & \exp \left[-a_{13}y - \frac{a_{22}}{y} \{ a_{23}y + (x + a_{13}y)z \} \right] z^n \\ & \quad L_n^\alpha \left[\frac{1}{yz} [a_{23}y + (x + a_{13}y)z], a_{12} + y + a_{22}z, a_{12} + y + a_{22}z \right] \\ & = \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} (-1)^{m+p} (n-k+l)_m (n+1)_l \\ & \quad L_{n+l-p}^{\alpha-k-l+m+p} [x, a_{12} + y + a_{22}z] y^{-k-l+m+p} z^{l-p+n} \end{aligned} \tag{1.1}$$

and by the following method of Weisner (Weisner 1955), the result (1.1) is obtained by applying the operator $e^{a_{23}A_{23}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} e^{a_{11}A_{11}} L_n^\alpha(x, y) z^n$. The object of this paper is to derive the possible variants of (1.1) by changing the order of A_{23}, A_{12}, A_{22} and A_{11} in $e^{a_{23}A_{23}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} e^{a_{11}A_{11}}$ by this change we have found the following new generating relations:

$$\begin{aligned} & \exp \left[-\frac{a_{22}xz}{y} - a_{13}(a_{12} + y + a_{22}z) \right] z^n \cdot L_n^\alpha \left[\frac{1}{yz} (a_{12} + y + a_{22}z), a_{12} + y + a_{22}z \right] \\ & = \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} (n-p+l)_l (n+m-k+l)_k \\ & \quad L_{n-p+l}^{\alpha-m+p-k-1} (x, y) y^{m+p-k-1} z^{l-p} \end{aligned} \tag{1.3}$$

$$\begin{aligned} & \exp \left[-\left\{ \frac{a_{23}}{y} - (xz + a_{23}y) + a_{12} + y + a_{22}z \right\} \right] z^n \cdot L_n^\alpha \left[\frac{1}{yz} (a_{12} + y + a_{22}z) \{ (x + a_{13}y)z + a_{23}y \}, a_{12} + y + a_{22}z \right] \\ & = \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{l=0}^{\infty} \frac{(a_{23})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} (n+m-k+l)_k (n+l)_l \\ & \quad L_{n+l-p}^{\alpha+m-k+1+p} (x, y) y^{m-p-k-1} z^{l-p} \end{aligned} \tag{1.4}$$

$$\begin{aligned} & \exp \left[-a_{13}y - \frac{a_{22}z}{y} (x + a_{13}y) \right] z^n \cdot L_n^\alpha \left[\frac{1}{yz} (a_{12} + y + a_{22}z) \{ a_{23}y + z(x + a_{13}y) \}, a_{12} + y + a_{12}z \right] \\ & = \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} (-1)^{m+p} (n-p+l)_l (n-k+l)_k \\ & \quad L_{n-p+1}^{\alpha+p-k+1+p} (x, y) y^{p-k-1+m} z^{l-p} \end{aligned} \tag{1.5}$$

We like to point out the following six operators:

$$\begin{aligned}
 & e^{a_{23}A_{23}} e^{a_{13}A_{13}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} \\
 & e^{a_{23}A_{23}} e^{a_{13}A_{13}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} \\
 & e^{a_{23}A_{23}} e^{a_{22}A_{22}} e^{a_{13}A_{13}} e^{a_{12}A_{12}} \\
 & e^{a_{13}A_{13}} e^{a_{23}A_{23}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} \\
 & e^{a_{13}A_{13}} e^{a_{23}A_{23}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} \\
 & e^{a_{13}A_{13}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} e^{a_{22}A_{22}}
 \end{aligned} \tag{A}$$

When applied to $L_n^\alpha(x, y)z^n$ will given rise to the result (1.1).

On the other hand the following six operators

$$\begin{aligned}
 & e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} e^{a_{13}A_{13}} \\
 & e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{13}A_{13}} e^{a_{23}A_{23}} \\
 & e^{a_{22}A_{22}} e^{a_{23}A_{23}} e^{a_{12}A_{12}} e^{a_{13}A_{13}} \\
 & e^{a_{12}A_{12}} e^{a_{22}A_{22}} e^{a_{23}A_{23}} e^{a_{13}A_{13}} \\
 & e^{a_{12}A_{12}} e^{a_{22}A_{22}} e^{a_{13}A_{13}} e^{a_{23}A_{23}} \\
 & e^{a_{12}A_{12}} e^{a_{13}A_{13}} e^{a_{22}A_{22}} e^{a_{23}A_{23}}
 \end{aligned} \tag{B}$$

When applied to $L_n^\alpha(x, y)z^n$ will gives rise to the result (1.3).

Again the following six operators

$$\begin{aligned}
 & e^{a_{23}A_{23}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{13}A_{13}} \\
 & e^{a_{23}A_{23}} e^{a_{12}A_{12}} e^{a_{13}A_{13}} e^{a_{22}A_{22}} \\
 & e^{a_{23}A_{23}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} e^{a_{13}A_{13}} \\
 & e^{a_{12}A_{12}} e^{a_{23}A_{23}} e^{a_{13}A_{13}} e^{a_{22}A_{22}} \\
 & e^{a_{12}A_{12}} e^{a_{23}A_{23}} e^{a_{22}A_{22}} e^{a_{13}A_{13}} \\
 & e^{a_{12}A_{12}} e^{a_{13}A_{13}} e^{a_{23}A_{23}} e^{a_{22}A_{22}}
 \end{aligned} \tag{C}$$

when applied to $L_n^\alpha(x, y)z^n$ will gives rise to the result (1.4).

Lastly the following six operations:

$$\begin{aligned}
 & e^{a_{13}A_{13}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} \\
 & e^{a_{13}A_{13}} e^{a_{22}A_{22}} e^{a_{23}A_{23}} e^{a_{12}A_{12}} \\
 & e^{a_{13}A_{13}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} e^{a_{23}A_{23}} \\
 & e^{a_{22}A_{22}} e^{a_{23}A_{23}} e^{a_{13}A_{13}} e^{a_{12}A_{12}} \\
 & e^{a_{22}A_{22}} e^{a_{13}A_{13}} e^{a_{12}A_{12}} e^{a_{22}A_{22}} \\
 & e^{a_{22}A_{22}} e^{a_{13}A_{13}} e^{a_{23}A_{23}} e^{a_{12}A_{12}}
 \end{aligned} \tag{D}$$

when applied to $L_n^\alpha(x, y)z^n$ will give rise to the result (1.5)

2. DERIVATIVE OF NEW GENERATING FUNCTION

From[11] (El-khazendar, 2013) we notice that:

$$A_{12} \left[L_n^\alpha(x, y)z^n \right] = nL_n^\alpha(x, y)y^{-1}z^n \tag{2.1}$$

$$A_{13} \left[L_n^\alpha(x, y)z^n \right] = \left(\frac{n}{x} - 1 \right) y L_n^\alpha(x, y)z^n - \frac{(\alpha + n)}{x} L_{n-1}^\alpha(x, y)z^n \tag{2.2}$$

$$A_{22} \left[L_n^\alpha(x, y)z^n \right] = -2(\alpha + n)L_{n-1}^\alpha(x, y)z^{n-1} \tag{2.3}$$

$$A_{23} [L_n^\alpha(x, y)z^n] = \left[\frac{n}{x} L_n^\alpha(x, y) - \frac{(n-\alpha)}{x} L_{n-1}^\alpha(x, y) \right] yz^{n-1} \tag{2.4}$$

where $A_{12} = xy^{-1} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $A_{13} = y \frac{\partial}{\partial x} - y$;

$$A_{22} = xy^{-1}z \frac{\partial}{\partial x} - z \frac{\partial}{\partial y} - xy^{-1}z;$$

$$A_{23} = yz^{-1} \frac{\partial}{\partial x}$$

Also the group corresponding to A_{12}, A_{13}, A_{22} and A_{23} are given by [11] (El-khazendar, 2013)

$$e^{a_{12}A_{12}}u(x, y, z) = u\left(\frac{x}{y}(a_{12} + y), a_{12} + y, z\right) \tag{2.5}$$

$$e^{a_{13}A_{13}}u(x, y, z) = e^{-a_{13}y}u(x + a_{13}y, y, z) \tag{2.6}$$

$$e^{a_{22}A_{22}}u(x, y, z) = e^{-\frac{a_{22}xz}{y}}u\left(x + a_{22}\frac{xz}{y}, y + a_{22}z, z\right) \tag{2.7}$$

$$e^{a_{23}A_{23}}u(x, y, z) = u\left(x, a_{23}\frac{y}{z}, y, z\right) \tag{2.8}$$

Now in order to prove (1.3) we can choose without any loss of generality the following operator from the set (B)

$$e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} e^{a_{13}A_{13}}$$

In fact we have

$$\begin{aligned} & e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} e^{a_{13}A_{13}} \left(L_n^\alpha(x, y)z^n \right) \\ &= e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} \sum_{m=n}^{\infty} \frac{(a_{12})^m}{m!} (-1)^m L_n^{\alpha+n}(x, y) y^m z^n \\ &= e^{a_{22}A_{22}} e^{a_{12}A_{12}} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} L_{n-p}^{\alpha+m+p}(x, y) y^{m+p} z^{n-p} \\ &= e^{a_{22}A_{22}} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} L_{n-p}^{\alpha+m+p-k}(x, y) y^{m+p-k} (n+m-k+1)_k z^{n-p} \\ &= \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} (n-p+l)_l (n+m-k+l)_k \\ & \quad L_{n-p+l}^{\alpha+m+p-k-l}(x, y) y^{m+p-k+l} z^{n-p+l} \end{aligned} \tag{2.9}$$

In other hand we have,

$$\begin{aligned} & e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} e^{a_{13}A_{13}} \left(L_n^\alpha(x, y)z^n \right) \\ &= e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} \left[e^{-a_{13}y} L_n^\alpha(x + a_{13}y, a_{12} + y + a_{22}z) \right] z^n \\ &= e^{a_{22}A_{22}} e^{a_{12}A_{12}} \left[e^{-a_{13}y} L_n^\alpha\left(x + a_{23}\frac{y}{z} + a_{23}y, a_{12} + y + a_{22}z\right) \right] z^n \end{aligned}$$

$$\begin{aligned}
 &= e^{a_{22}A_{22}} \left[e^{-a_{13}y} (a_{12} + y) \cdot L_n^\alpha \left\{ (a_{12} + y) \left(\frac{x}{y} + \frac{a_{23}}{z} + a_{13} \right), a_{12} + y + a_{22}z \right\} \right] (a_{12} + y) z^n \\
 &= e^{a_{22} \frac{xz}{y} - a_{13}(a_{12} + y + a_{22}z)} \cdot L_n^\alpha \left[\frac{1}{yz} (a_{12} + y + a_{22}z) \{ a_{13}y + z(x + a_{13}y) \}, a_{12} + y + a_{22}z \right] (a_{12} + y + a_{22}z) z^n
 \end{aligned}
 \tag{2.10}$$

Using (2.9) and (2.10) we get

$$\begin{aligned}
 &\exp \left[a_{22} \frac{xz}{y} - a_{13}(a_{12} + y + a_{22}z) \right] \cdot (a_{12} + y + a_{22}z) z^n \\
 &L_n^\alpha \left[\frac{1}{yz} (a_{12} + y + a_{22}z) \{ a_{13}y + z(z + a_{13}y) \}, a_{12} + y + a_{22}z \right] \\
 &= \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} (n - p + l)_l (n + m - k + l)_k \\
 &L_{n-p+l}^{\alpha+m+p-k-l}(x, y) y^{m+p-k-l} z^{l-p}
 \end{aligned}$$

which is (1.3).

Now in order to prove (1.4) we can choose without ant loss of generality following operator from the set (C)

$$e^{a_{23}A_{23}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{13}A_{13}} .$$

Hence the calculation shown in deriving (1.3) is a routine one, we only mention main steps in deriving (1.4) and (1.5).

We have

$$\begin{aligned}
 &e^{a_{23}A_{23}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{13}A_{13}} \left(L_n^\alpha(x, y) z^n \right) \\
 &= \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} (n + m - k + l)_k (n + l)_l \\
 &L_{n-p+l}^{\alpha+m+p-k-l}(x, a_{12} + y + a_{22}z) y^{m-k-l+p} z^{n-p+l}
 \end{aligned}
 \tag{2.11}$$

On the other hand we have:

$$\begin{aligned}
 &e^{a_{23}A_{23}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{13}A_{13}} \left(L_n^\alpha(x, y) z^n \right) \\
 &= \exp \left[- \left\{ \frac{a_{22}}{y} (xz + a_{22}y) + a_{13}(a_{12} + y + a_{22}z) \right\} \right] \cdot z^n (a_{12} + y + z) \\
 &L_n^\alpha \left[\frac{1}{yz} (a_{12} + y + a_{22}z) \{ a_{23}y + z(x + a_{13}y) \}, a_{12} + y + a_{22}z \right]
 \end{aligned}
 \tag{2.12}$$

Equating (2.11) and (2.12) we get.

$$\begin{aligned}
 &\sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} (-1)^{m+p} (n + m - k + l)_k (n + l)_l \\
 &L_{n+1-p}^{\alpha+m-k+l+p}(x, y) z^{l-p} y^{m-k-l+p}
 \end{aligned}$$

which is (1.4).

Lastly in order to prove (1.5) we can choose without any loss of generality the following operator from the set (D)

$$e^{a_{13}A_{13}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} .$$

We have

$$\begin{aligned} & e^{a_{13}A_{13}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} \left(L_n^\alpha(x, y) z^n \right) \\ &= \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} (-1)^{m+p} (n-k+l)_k (n-p+l)_l \end{aligned} \quad (2.13)$$

On the other hand we have

$$\begin{aligned} & e^{a_{13}A_{13}} e^{a_{22}A_{22}} e^{a_{12}A_{12}} e^{a_{23}A_{23}} \left(L_n^\alpha(x, y) z^n \right) \\ &= \exp \left[-a_{13}y - a_{22} \frac{z}{y} (x + a_{13}y) \right] (a_{12} + y + a_{22}z) \cdot z^n \\ & \quad L_n^\alpha \left[\frac{1}{yz} (a_{12} + y + a_{22}z) \{ a_{23}y + z(x + a_{13}y) \}, a_{12} + y + a_{22}z \right] \\ &= \sum_{m=0}^{\infty} \frac{(a_{13})^m}{m!} \sum_{l=0}^{\infty} \frac{(a_{22})^l}{l!} \sum_{k=0}^{\infty} \frac{(a_{12})^k}{k!} \sum_{p=0}^{\infty} \frac{(a_{23})^p}{p!} (-1)^{m+p} (n-k+l)_k (n-p+l)_l \\ & \quad L_{n-p+1}^{\alpha+p-k+l+p}(x, y) y^{p-k-1+m} z^{l-p} \end{aligned} \quad (2.14)$$

which is (1.5).

REFERENCES:

- [1] S.K. Chatterjea, (1975): Group theoretic arights of certain generating functions of Laguerre polynomials. *Bull. Inst. Math. Acad. Sinica*, 3 (2) 369-375.
- [2] C.C. Feng, (1978): some generating functions of modified Laguerre polynomials. *Hokkaido math. Jour.* 7, 189.
- [3] E.B. McBride, (1971) *Obtaining generating functions*, springer-verlag. Berlin.
- [4] L. Weisner, (1955) Group theoretic origins of certain generating functions, *Pacific J. Math.*, 5, 1033-39.
- [5] L. Weisner, (1959) Generating functions for Hermite functions, *Canad. J. Maths*, II, 11,141-147.
- [6] L. Weisner (1959) Generating functions for Bessel functions, *Canad. J. Maths*, II, 11,148-155.
- [7] W. Miller (1968) *Lie theory and special functions*, Academic Press. New York and London
- [8] H.M. Srivastava, and H.L. Monocha, (1955) A treatise on generating functions, *Pacific J. Maths.* 1033-1039.
- [9] G. Dottoli, (2000) Generalized polynomials, operational Identities and their applications. *Journal of Computational and Applied mathematics*, 118, 111-123.
- [10] S. Khan, and G. Yasmin, (2003) Lie theoretic generating relations of two variables Laguerre polynomials, *Reports on Mathematical Physics*, 52, 1-7.
- [11] M. B Elkhazendar, (2013) Some generating functions of two variables Laguerre polynomials from the view point of Lie Algebra. *Gen. Math. Notes.* 14 (2) pp53-62.

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