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MODELING OF SOME CONCEPTS FROM NUMBER THEORY TO GROUP THEORY

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ABSTRACT

In this paper, we extended the notion of multiplicative numbers and superperfect numbers to finite groups. We provide some general theorem and present examples of multiplicative groups and superperfect groups. Also, we prove some related results.

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1. INTRODUCTION

The study of perfect numbers has been in progress for as long as many other important mathematical fields ([3, 4, 5, 6, 7, 8]). Although it is unknown when the study of perfect numbers first began, there is clear evidence of perfect numbers being studied as early as Pythagoras. For any positive number n, we define $\sigma(n)$ the sum of the divisors of n that σ is multiplicative and n is called perfect if $\sigma(n) = 2n$, for example 6 is perfect, since 6 = 1+2+3+6=12 ([9]). Also, n is called superperfect if $\sigma(\sigma(n)) = 2n$ ([11]) and called deficient if $\sigma(\sigma(n)) < 2n$. Leinster in [2] extended the notion of perfect numbers to finite groups. He called a finite group is perfect (FPG) if its order is equal to the sum of the orders of all normal subgroups of the group. In the other words, G is called perfect group if $\sigma(G) = \sum_{N \trianglelefteq G} |N| = 2|G|$, for example C_6. Here we go over the basic properties of multiplicatively perfect numbers. Let T(n) denote the product of all divisors of n. There are many numbers n with the property $T(n)=n^2$, but none satisfying $T(T(n))= n^2$. Let us call the number n>1 multiplicatively perfect if $T(n)= n^2$, and n is a deficient multiplicatively perfect if $T(n) < n^2$. ([1, 5] for other type of numbres).

Now, in this paper we define multiplicative and superperfect groups and we prove some related results.

Theorem 1.1: (Euclid) If $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect. ([1])

Definition 1.2: (Mersenne Numbers) When $2^p - 1$ is prime, it is called a mersenne prime.

Theorem 1.3:
(i) Even superperfect numbers are $2^p - 1$, where $2^p - 1$ is a mersenne prime.
(ii) If any odd superperfect numbers exist, they are square numbers and either n or $\sigma(n)$ is divisible by at least three distinct primes. ([10])

Definition 1.4: Let $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$ such that $\sigma^0(n) = n$. We call n is (m, k)-perfect if $\sigma^m(n) = kn$.

Notice: The classical perfect numbers are (1, 2)-perfect. Multiperfect numbers are (1, k)-perfect and superperfect numbers are (2, 2)-perfect.

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2. MODELING OF SOME CONCEPTS FROM NUMBER THEORY TO GROUP THEORY

Definition 2.1: Let G be a finite group such that $\sigma(\sigma(G)) = 2|G|$. Then G is said to be superperfect group. ([12])

Example 2.2: Let $Q_8 = < a,b : a^4 = e, a^2 = b^2, ba = a^{-1}b>$ be a quaternion group of order eight and H = < e, a, a^2, a^3 > be a normal subgroup of Q_8. Then H is a superperfect group, because $\sigma(\sigma(H)) = 8 = 2|H|$.

Example 2.3: Let D_8 be a dihedral group of order eight. And H = < e, a, a^2, a^3 > be a normal subgroup of D_8. Then H is a superperfect group because $\sigma(\sigma(H)) = 8 = 2|H|$.

Proposition 2.4: Let G be an abelian finite group of order 2 or 4. Then G is a superperfect group.

Proof: First assume that $|G| = 2$ so $G \cong Z_2$. Therefore, $\sigma(\sigma(G)) = 4$. Now, let $|G| = 4$ then $G \cong Z_4$ or $G \cong Z_2 \times Z_2$. Therefore, $\sigma(\sigma(G))$ =8. The proof is finished.

Theorem 2.5: Let G be a FPG such that $|G| \neq 1$. Then G is not superperfect group.

Proof: Let G be a FPG with $|G| = n$ ($n \neq 1$). Therefore, we have $\sigma(G) = 2|G|$ =2n. But $\sigma(\sigma(G)) = \sigma(2n) \neq 2n$. Therefore, G is not superperfect group.

Question 2.6: Is it true to say that, if G be a FPG and H be a nontrivial subgroup of G then H is not FPG?

Definition 2.7: Let G be a finite group, we define $T(G)=\prod_{N \trianglelefteq G} |N|$, the product of the orders of the normal subgroups of G, and say that G is multiplicatively perfect if $T(G)=|G|^2$.

Example 2.8: (Cyclic Groups) Let C_n be the cyclic group of order n. Then C_n has one normal subgroup of order d for each divisor d of n, and no others. So $T(C_n)=T(n)$ and C_n is multiplicatively perfect just when n is m-perfect. Thus multiplicatively perfect groups provide a generalization of the concept of perfect numbers and C_6, C_{15}, C_{21}, C_{22}, C_{26},... are all multiplicatively perfect groups.

Corollary 2.9: Let C_6 be the cyclic group of order 6. This group is the only perfect group, which is also multiplicatively perfect.

Example 2.10: (p-Groups) A finite abelian p-group is a group of order P^r, where p is prime and r $\geq$0. Lagrange's theorem says that the order of any subgroup of a group divides the order of the group. So if G is an abelian p-group of order P^r, where r $\geq$ 0 and r$\neq$3, then G is not multiplicatively perfect.

Corollary 2.11: The only abelian p-group of order P^3 is a multiplicatively perfect group.

Example 2.12: (Symmetric and Alternating Groups) For n $\leq$ 4, we have
$T(A_1) = 1$; $T(S_1) = 1$
$T(A_2) = 1$; $T(S_2) = 2$
$T(A_3) = 1 \times 3 = 3$; $T(S_3) = 1 \times 3 \times 6 = 18$
$T(A_4) = 1 \times 4 \times 12 = 48$; $T(S_4) = 1 \times 4 \times 12 \times 24 = 1152$.

So if n $\leq$ 4, then A_1 and S_1 are the only groups that are multiplicatively perfect.

Example 2.13: (Dihedral Groups) let E_{2n} be the dihedral group of order 2n. That is, the group of all isometries of a regular n-sided polygon. Of the 2n isometries, n are rotations (forming a cyclic subgroup of order n) and n are reflections. We examine the cases of n odd and n even separately. If **n odd**: All reflections are in an axis passing through a vertex and the midpoint of the opposite side, and any reflection is conjugate to any other by a suitable rotation. Thus if $N \trianglelefteq E_{2n}$ and N contains a reflection, that N contains all reflections; but 1 into N too so |N|$\neq$n+1, so N =E_{2n}. So any proper normal subgroup is inside the rotation group C_n. Conversely, any (normal) subgruop of C_n is normal in E_{2n}. Thus $T(E_{2n}) = T(C_n)\times 2n$, and E_{2n} is not multiplicatively perfect group. If **n even**: The reflections split in to two conjugacy classes, R_1 and R_2 which each of size $\frac{n}{2}$. Write $C_{n/2}$ for the group of rotations by 2 or 4 or ... or n vertices. A subgroup of E_{2n} which is cyclic of order of $\frac{n}{2}$. Then we can show that smallest subgroup of E_{2n} contaning R_i is $R_i+C_{n/2}$, for i=1 and 2. Moreover, $R_i+C_{n/2}$ is of order n, i.e. index 2. Therefore, $R_i+C_{n/2}$ is normal in E_{2n}. So we

have two different normal subgroups, $R_1+C_{n/2}$ and $R_2+C_{n/2}$, of order n. We also have the normal subgroups 1 and E_{2n} . Hence $T(E_{2n})\neq 1\times n\times n\times 2n$, and E_{2n} is multiplicatively perfect if and only if n = 2.

Theorem 2.14: Let G_1 and G_2 be two finite groups of order P_1 and P_2 respectively, where P_1 and P_2 are distinct prime numbers. Then $T(G_1 \times G_2) = T(G_1)^{\varphi(|G_2|)-1} T(G_2)^{\varphi(|G_1|)-1}$, such that φ(n) is the number of the divisors of n.

Proof: The result follows by definition.

Definition 2.15: Let G be a finite group. Then $\Delta(G)=\frac{T(G)}{|G|}$.

Observation 2.16: For any normal subgroup N of any finite group we have
(i) If $G \neq 1$, then $\Delta(G/N) \leq \Delta(G)$.
(ii) G is simple group if and only if $\Delta(G) = 1$.

Theorem 2.17: If G is finite nilpotent group with $T(G) \leq |G|^2$, then G is cyclic and |G| is a multiplicatively perfect or multiplicatively deficient number.

Proof: Let G be a finite nilpotent group with $T(G) \leq |G|^2$ and let H be an arbitrary Sylow p-subgroup of G for some prime divisor p of |G|. In particular, H is a direct factor of G. By observation 2.16 (i), it followes that $T(H) \leq |H|^2$. Now, let F denote the Frattini subgroup of H, then $\frac{H}{F}$ is an elementary abelian p-group whose rank, say r, is the minimal number 3 of generators of H. By observation 2.16 (i), we have $T(\frac{H}{F}) \leq |\frac{H}{F}|^2$. On the other hand, if r > 1, then $p \leq T(\frac{H}{F})$ since $\frac{H}{F}$ contains at least p+1 subgroups of order P^{r-1}. This forces r = 1. Hence H is cyclic. Then every Sylow subgroup of G is cyclic. Since G is nilpotent, this implies that G itself must be cyclic. Therefore, T(G) = T(|G|), and we conclude that |G| must be a multiplicatively perfect or a multiplicatively deficient number.

Corollary 2.18: Every nilpotent quotient of a multiplicatively perfect group is cyclic.

Lemma 2.19: Let G be a finite group and p be a prime, then the number of normal subgroups of G with index p is $\frac{P^r - 1}{p-1} = 1 + p + \ldots + P^{r-1}$; for some $r \geq 0$.

Proof: See the Lemma 5.1 of [2].

Definition 2.20: Let G be a finite group. Then G is called a tight if for each prime p, G has at most one normal subgroup of index p.

Proposition 2.21:
(i) A group G with $T(G) \leq |G|^2$ is tight.
(ii) A quotient of a tight group is tight.
(iii) A tight abelian group is cyclic.

Proof:

(1) (i) For each prime p, we have $|G|^2 \geq T(G) \geq |G| \frac{P^r - 1}{p-1} . \frac{|G|}{p}$, where r is as in Lemma 2.19.
If $r \geq 2$ then $\frac{P^r - 1}{p-1} . \frac{|G|}{p} \geq (p+1) \frac{|G|}{p} > |G|$ is a contradiction. Thus r is 0 or 1, and so $\frac{P^r - 1}{p-1}$ is 0 or 1.

(2) (ii) Let $\pi : G_1 \rightarrow G_2$ be a surjective homomorphism. If N and N' are distinct normal subgroups of G_2 with index p, then $\pi^{-1} N$and $\pi^{-1} N'$ are distinct normal subgroups of G_1 with index p.

(3) (iii) For this we invoke the classification theorem for finite abelian groups, which tells us that for any abelian group A there exist primes $P_1, \ldots, P_n$ and numbers $t_1, \ldots, t_n \geq 1$ such that $A \cong Z_{{P_1}^{t_1}} \times Z_{{P_2}^{t_2}} \times \ldots \times Z_{{P_n}^{t_n}}$. Suppose that $P_i = P_j$ (= p, say) for some $i \neq j$. Then, since $t_i \geq 1$, $Z_{p^{t_i}}$ has a (normal) subgroup N_i of index p and similarly $Z_{p^{t_j}}$. Hence $N_i \times Z_{p^{t_j}}$ and $Z_{p^{t_i}} \times N_j$ are distinct index-p subgroups of $Z_{p^{t_i}} \times Z_{p^{t_j}}$ and $Z_{p^{t_i}} \times Z_{p^{t_j}}$ is not tight. Since $Z_{p^{t_i}} \times Z_{p^{t_j}}$ is a quotient of A, part (ii) implies that A is not tight either. Thus if A is tight then all the P_k are distinct, so that $A \cong Z_{{P_1}^{t_1} {P_2}^{t_2} \ldots {P_n}^{t_n}}$.

Definition 2.22: An abelian quotient of a group G is just a quotient of G which is abelian. That is, it's an abelian group A for which there exists a surjective homomorphism $G \rightarrow A$, alternatively, it's an abelian group isomorphic to G/K for some normal subgroup K of G.

Theorem 2.23: (Abelian Quotient Theorem) If G be a finite group with $T(G) \leq |G|^2$ then any abelian quotient of G is cyclic.

Proof: Putting together the three parts of the last proposition gives us the proof of the abelian quotient theorem.

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