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BALANCED GROUPS AND RINGS

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ABSTRACT

A finite group G is balanced if the product of its elements, in some order, or its square is the identity element. We give examples of some balanced groups. Let R be a ring (not necessarily commutative) with a finite cyclic group U of units of order n. It is shown that the product of units is -1 if n is even and it is 1 if n is odd. Let D be an integral domain with a finite number n of units and with ch(D) different from 2. It is shown that the product of units is -1. Let R be a commutative ring with 1 whose characteristic is not 2 and that has a finite group of units U and let the order of U be n. It is shown that n is even. Let R be a commutative ring with 1 which has a finite number of units and let A be the nil radical of R. Then R/A has a finite number of units whose product is 1 or -1 mod A.

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INTRODUCTION

A finite group is called *n-balanced* (NB) if we can arrange the elements in some order such that when we multiply out these elements in this order and raise the result to the n-the power we get the identity. The group is *freely n-balanced* (FNB) if the product of its elements, in any order, when raised to the n-the power gives the identity. A ring with unity is called NB or FNB if it has NB or FNB finite group of units. Let us call a group or a ring to be *balanced* if it is 1 or 2-balanced, free or not. Let G be a multiplicative group of odd order 2n+1. Then G is 1B: For there is no elements which coincides with its own inverse except the identity and so we get $e(b_1 b_1^{-1})\ldots(b_n b_n^{-1}) = e$. Thus in any finite field F of characteristic 2 the non zero elements form a 1B multiplicative group. Let G be a multiplicative cyclic group of even order 2m then it is F2B. For, the product of its elements is the unique element of order 2. It follows that if p is an odd prime and if F is a finite field of characteristic p then the group of units is F2B. By the fundamental theorem of finite Abelian groups every finite Abelian group is F1B or F2B.

Proposition 1: Any finite product of balanced groups (or rings) is a balanced (group) or ring.

The proof is trivial.

Example 1:

(a) The symmetric group S_3 is 2-balanced. For (123)(12)(13)(23)(132)=(123)(13)(132)=(12) and (12) has order 2. We notice that the equilateral triangle is balanced.

(b) Similarly the dihedral group $D_4 = \{e, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$ is 2-balanced: for the product of the first four elements is (13)(24), the product of the last four elements is e and the square of (13)(24) is e. Again the square is balanced. It looks like that the general dihedral group D_n is balanced although I have no proof for this.

(c) It can be shown that S_4, A_4, V_4 are balanced.

Question 1: For every n is it true that S_n, A_n are balanced?

Question 2: Is every finite non Abelian group is balanced?

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We restrict our attention in the rest of this paper to a special class of groups, namely the group of units of a ring and especially commutative rings.

Remark 1: (1) Let U be the group of units of the ring Z_{p^n}, where p is an odd prime and $n \geq 1$. Then U is F2B.

(2) Let U be the group of units of Z_{2^n}, n>1. Then U is F2B.

(3) Let U be the group of units of Z_{2p^n}, where p is an odd prime and $n \geq 1$. Then U is F2.

Proof:

(1) Assume p is odd. Then the group of units is of even order $p^n - p^{n-1}$ and it is cyclic because the number p^n has a primitive root. Then from proposition 1 the product is -1.

(2) If n=1 then the product of units is 1. If n=2 then the product of units is -1. Let $n \geq 3$ then the multiplicative group of units is isomorphic with the additive group G= $Z_2 \oplus Z_{2^{n-2}}$ and so the product of units corresponds to the sum of elements in G which is 0 and so the product of units in R is 1.

(3) Let U be the multiplicative group of units of R. Then U is isomorphic to the additive cyclic group $Z_{p^n - p^{n-1}}$ =S= Z_{2m}. Now the product b of elements of U corresponds to the sum a of elements of group S and a=m of order 2 and it is the only such element. Since -1 is in U and it is of order 2 it follows that the product of units is-1.

Proposition 2: Let D be an integral domain with a finite group U of units of order n. Then D is balanced.

Proof: If the characteristic of D is 0 then D contains a copy of Z and the units of D form a non trivial subgroup of the cyclic group of some n-th roots of unity. By proposition 1 the product of units is 1 or -1. If the characteristic is p then the group of units U(D)is a subgroup of the cyclic group of non zero elements of a finite field F of characteristic p. Then by proposition 1 the product of units is 1 or -1.

Remark 2:

(1) Let p be a prime integer and let $R = Z_{p^n}$. Let U be the group of units of R. Let b be the product of elements of U. Then $b = -1$. We have two cases to consider: p=2 and p $\neq$ 2.

Let p=2. If n=1 then b=1=-1. If n=2 then the b=3=-1. If n>2 then multiplicative group U is isomorphic with the additive group G= $Z_2 \oplus Z_{2^{n-2}}$. Then the product b of elements of U corresponds to the sum of elements of G which is (0,0) and so b=1=-1.

Now we consider the case p is an odd prime. Then p^n has a primitive root. Thus U is cyclic of even order $p^n - p^{n-1}$ and so b=-1 by corollary 1.

(2) Let $R = Z_{2p^n}$, p is an odd prime. Then $2p^n$ has a primitive root and so U is cyclic of even order and so the product, by corollary 1 is b=-1. The last assertion is clear since the radical is isomorphic with Z_2.

Proposition 3: Let R be the ring $Z_m, m = p_1^{e_1} \ldots p_n^{e_n}$,n>1, $e_i > 1$,i=1,...,n, where the p_i are distinct primes. Let U be the multiplicative group of units in R and let b be the product of elements of U.
Then b=1 in R.

Proof: The ring R is isomorphic with the ring $T = R_1 \oplus \ldots \oplus R_n$, where $R_i = Z_{p^{e_i}}$. Let U_i=U(R_i), k_i the order of U_i and let b_i be the product of elements of U_i. Let l_i be the sum of all k_j except k_i. Then U(R) is isomorphic with the direct sum of U_i. If p_1 is not 2 then all l_i are even and b is the product of all $b_i^{k_i}$ which is equal to (1,...,1) =1_R . If p_1=2 e_1=e>1. Then R=Z $\oplus$ S, S is nontrivial. Then b is the product of even number of the element b′= (±1, 1) and this is 1_R.

Proposition 4: Let R be a commutative ring with 1 whose characteristic is not 2 and which has a finite group U of units of n elements. Then n is even.

Proof: Assume that U has odd number of elements. Then U is isomorphic to a finite direct product of n primary multiplicative cyclic groups C_i=<a_i >,i=1,…n each of order some prime power. Let us first assume n=1 and so U=<a>

and let the order of a be m. Now -1 is a unit different from 1 and so $-1=a^k$. But then $a^{2k}=1$. Using division algorithm we have $2k=lm+r$, $0\leq r<n$. It follows that $a^r=1$ and this is a contradiction since $0\leq r<n$ .In case n=2 then U is the direct sum of two cyclic groups $<a_1>$, $<a_2>$ of order k, l respectively. Then -1 can be represented as a pair $(a_1^{k'}, a_2^{l'})$. Thus 1 corresponds to $(a_1^{2k'},a_2^{2l'}) = (1,1)$. Using division algorithm we get a similar contradiction. The general case follows from this.

Example 2: The ring Z_8 has a group of units of 4 elements and their product is 1. Let G=<a>, a^3. Let R=b $Z_2[G]$ be group ring of Z_2 over G. Then R is a commutative ring with 8 elements. Its characteristic is 2 and the group G of units has 3 elements with a product 1. In most cases the product of units was always 1 or -1. This is not always true in commutative rings with a finite group of units.

Example 3: Let $G=<x: x^4 =1>$ be a multiplicative cyclic group of order 4.

Let $R=Z_2[G]$ be the group ring of G over Z_2. Then R has 16 elements of the form $a+bx+cx^2+dx^3$,a,b,c,d are in Z_2. Let V be the multiplicative group of units of R. Then V is contained in the set $W=\{1+ax+bx^2+cx^3$,a,b,c are in $Z_2\}$. Now W has 8 elements: A=1,B=1+x,$C=1+x^2$,$D=1+x^3$,$E=1+x+x^2$, $F=1+x+x^3$,$G=1+x^2+x^3$, $H=1+x+x^2+x^3$. Since $C^2=0$ and $H^2=0$, the possible units are A,B,D,E,F,G. Since A=1 we have to check only the elements B, D, E, F, G. Here is the multiplication table for these elements:

.	B	D	E	F	G
B	C	xC	D	xC	xB
D	xC	C	x^2B	xB	B
E	D	x^2B	x^3	x	0
F	xC	xB	x	1	x^3
G	xB	B	0	x^3	x^2

We see that the only units in R are 1 and F and so the number of units is 2. The product of units is F which is neither 1 nor -1.

Let R be a ring. We denote by $\sqrt{R}$ the nil radical of the ring R generated by the set of nilpotent elements in R.

Remark 3: Let A be the nil radical of the ring R in the example above. Then we notice that the product of units is $F=1+(x+x^3)$ and $(x+x^3)^2=x^2+x^6=x^2+x^2=0$. Thus the product of units is 1 mod A.

The following is, in a way, a generalization of the remark above.

Proposition 5: Let R be a commutative ring with 1 which has a finite number of units and let A be the radical of R. Then R/A has a finite number of units whose product b = 1 or -1 mod A.

Proof: R/A is a commutative ring which has no nilpotent elements and let $U=\{u_1,\ldots,u_n\}$ be the group of units. Then from the fundamental theorem of Abelian groups $U=\oplus_{i=1}^{k}U_{n_i}$, n_i divides n_{i+1}, i=1,...,k-1 and U_{ni} is cyclic of order n_i. In our case each n_i is even. In case k=1 then U=<a> is cyclic of order 2n and the product of its elements is the element $b=a.a^2...a^{2n-1}$. The sum of the exponents in the left hand side b is ((1+2n-1)/2). $(2n-1)=2n^2-n=n$ mod 2n. Thus b has order 2. Since U is cyclic there is only one such element. Since -1 is a unit and has order 2 then $a^n=-1$. If k>1 then the product of units is the same as the product of units of order 2. Each such element is of the form $(x_1,...,x_k)$ where each x_i is either the identity element of U_{ni} or the unique element a_i in U_{ni} of order 2. Since k>1, when we multiply out the units each a_i is repeated even number of times. Thus the product is 1.Therefore the product of units in R/A is 1 or -1 mod A.

In the following we compute the product of units of other two examples of a commutative rings with a finite group of units

Example 4:

(1) Let m be a positive integer and let ω be a primitive m-th root of unity in the complex plane. Let R=Z[ω].If m is odd the group of units constitutes of the m-roots of 1 together with their negatives and so it is the group of the 2m-th roots of 1 and it is a cyclic group of order 2m. If m is even then -1 is an m-th root of 1 and we see that the set of m-th roots of 1 coincides with the group of units of R which is then cyclic and the product of units is -1.

(2) Let m be a positive integer and let R be be the ring Z equipped with all m-th roots of -1 in the complex plane. Then the group of units of R is the group of 2m-th roots of unity. We see that the product of units is -1.

So far, we have seen that the number of units in a commutative ring is an even integer. There is an open question:

Question 3: Given a finite Abelian group G of even order 2n is there a commutative ring R with 1 such that the group of units is isomorphic with U?.

We give an example of a finite non commutative ring with 1 whose group of units U is finite and isomorphic with S_3.

Example 5: Let R be the matrix ring of all 2 by 2 matrices over Z_2. There are 16 matrices in R and U is non Abelian that has 6 elements each of which has determinant 1. They are

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, c = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

U is isomorphic with S_3.

Proposition 6: Let U be a given finite Abelian group. There is a commutative ring R with 0-nilradical and with group of units containing U.

Proof: By the fundamental theorem of Abelian groups U is a direct sum of primary cyclic groups. Assume that the corresponding orders are $n_1,\ldots,n_k$, respectively. For every n_i we take the ring R_i to be the ring Z equipped with the set of all n_i-th roots of 1 and we take R to be the direct sum of the rings R_i. Clearly the group of units of R contains the group U.

We finish with

Question 4: Is any finite group balanced?

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