Research Journal of Pure Algebra -1(3), June - 2011, Page: 77-80
Available online through www.rjpa.info ISSN 2248 – 9037

X -GORENSTEIN INJECTIVE MODULES

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(Received on: 24-05-11; Accepted on: 09-06-11)

ABSTRACT

In this paper, we generalize the notion of Gorenstein injective modules. Namely, we introduce X -Gorenstein injective modules, where X is a class of modules that contains all injective modules. We show that the principal results on Gorenstein injective module remain true for the X -Gorenstein injective modules.

Mathematics Subject Classification: *13D02, 13D07*

Keywords: *Gorenstein injective modules, X -Gorenstein injective modules.*

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1. INTRODUCTION:

Throughout this paper, R denotes a non-trivial associative ring with identity, and all modules are left R-modules.

In 1967-69, Auslander and Bridger [1, 2] introduced the G-dimension for finitely generated R-modules when R is Noetherian, denoted by G – dim(M) where M is a finitely generated R-module. As the classical case, the G-dimension
of modules is defined in terms of resolutions by modules of G-dimension 0, which are defined as follows:

A finitely generated R-module M has G-dimension 0, if:

• Ext^m (M, R) = 0 = Ext^m (HomR (M, R), R) for every m > 0; andRR

• M is reflexive, that is, the canonical map M → Hom_R (Hom_R (M, R), R) is an isomorphism.

In [1], Auslander proved that a finitely generated R-module M has G-dimension 0 if and only if there exists an exact sequence of finitely generated free R-modules **L** = $\cdots \to L_1 \to L_0 \to L^0 \to L^1 \to \cdots$ such that M ~ $\mathrm{Im}(L_0 \to L^0)$=
and the complex Hom_R (L, R) is exact.

In [5, 6], Enochs and Jenda defined, over arbitrary rings, the Gorenstein injective modules as follows:

Definition: 1.1 An R-module M is said to be Gorenstein injective, if there exists an exact sequence of projective R-modules

I = $\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$

such that M ~ $\mathrm{Im}(I_0 \to I^0)$ and such that HomR (I, –) leaves the sequence **I** exact whenever I is a injective R-module.

The exact sequence **I** is called a complete injective resolution.

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Namely, we define X -Gorenstein injective modules, where X is a class of R-modules that contains all injective R-modules (see Definition 2.1). In Proposition 2.2 we characterize the X -Gorenstein injective modules. Our main result is Theorem 2.3, in which, we study the behavior of the notion of X -Gorenstein injective modules in short exact sequences.

2. X -GORENSTEIN INJECTIVE MODULES:

In this section we investegate the following generalization of Gorenstein injective modules.

Definition: 2.1 Let X be a class of R-modules that contains all injective R-modules. An R-module M is called X -Gorenstein injective, if there exists an exact sequence of injective R-modules **I** = $\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$ such that $M \sim \mathrm{Im}(I_0 \to I^0)$ and HomR (F, **I**) is exact whenever $F \in X$.

The sequence **I** is called an X -complete injective resolution.

We start with the following characterization of an X -Gorenstein injective module.

Proposition 2.2 For an R-module M , the following conditions are equivalent:

1. M is X -Gorenstein injective.

2. (i) $\mathrm{Ext}^i(F, M) = 0$ for every $F \in X$ and every $i > 0$;

   (ii) There exists an exact sequence of R-modules **I** = $\cdots \to I_0 \to I^1 \to M \to 0$, where each Ii is injective, such that $\mathrm{Hom}_R(F, \mathbf{I})$ is exact for every $F \in X$.

3. There exists a short exact sequence of R-modules $0 \to N \to I \to M \to 0$, where I is injective and N is X -Gorenstein injective.

4. There exists a family of short exact sequences of R-modules $0 \to M_{i+1} \to I_i \to M_i \to 0$ ($i \in Z$), where each Ii is injective and $M_0 = M$, such that

   $\mathrm{Ext}^1(F, Mi) = 0$ for every $F \in X$ and every $i \in Z$.

Proof: The proof of the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) is analogous to the ones of the Gorenstein injective counterpart (see [4, 7]).

The implication (3) $\Rightarrow$ (4) is obvious.

To end, we prove the implication (3) $\Rightarrow$ (2). Let $F \in X$. Applying the functor

HomR (F, –) to the exact sequence $0 \to N \to I \to M \to 0$, we get the long exact sequence: $\cdots \to \mathrm{Ext}_i(F, M) \to \mathrm{Ext}_i(F, I) \to \mathrm{Ext}_i(F, N) \to \cdots$ For every $i > 0$, we have: $\mathrm{Ext}_R(F, N) = 0$ (since N is X -Gorenstein injective and by the equivalent (1) $\Leftrightarrow$ (2)). Also, we have $\mathrm{Ext}_i(F, I) = 0$ (since I is injective).

Then, $\mathrm{Ext}_R(F, M) = 0$ for every $i > 0$.

It remains to prove (ii). Since N is X -Gorenstein injective and by the equivalent (1) $\Leftrightarrow$ (2), there exists an exact sequence of R-modules $I = \cdots \to I_0 \to I_1 \to N \to 0$, where each I_i is injective, such that $\mathrm{Hom}_R(F, I)$ is exact for all R-modules $F \in X$. Assembling this sequence with the short exact sequence $0 \to N \to I \to M \to 0$ we get the following exact sequence $E = \cdots \to I_1 \to I_0 \to I \to M \to 0$ such that the sequence $\mathrm{Hom}_R(F, E)$ is exact for every R-module $F \in X$, as desired.

The following result, which investigates the behavior of X -Gorenstein injective modules in short exact sequences, generalizes [7, Theorem 2.6].

Theorem: 2.3

(1) Let $0 \to C \to B \to A \to 0$ be a short exact sequences of R-modules, where C is X -Gorenstein injective. Then, A is X -Gorenstein injective if and only if B is X -Gorenstein injective.

(2) Let $(Mi)_{i\in I}$ be a family of R-modules. Then, M_i is X –Gorenstein injective if and only if M_i is X -Gorenstein injective for every $i \in I$.

Proof: The equivalences of both (1) and (2) can be proved similarly to the one of [7, Theorem 2.6]. Here, we give a new and simple proof of the "only if" part of (1). Then, assume that B is X -Gorenstein injective. By Proposition 2.2 (1) ⇔ (3), there exists an exact sequence of R-modules $0 \to G \to I \to B \to 0$, where I is injective and G is X -Gorenstein injective. Consider the following

Pullbacks diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G \arrow[d] \arrow[r, equal] & G \arrow[d] & & \\
0 \arrow[r] & C \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Applying the "if" part to the left vertical short exact sequence, we get that C is X -Gorenstein injective. Therefore, use the equivalence (1) ⇔ (3) of Proposition 2.2 and the middle horizontal short exact sequence to get that A is X -Gorenstein injective.

We end the paper with a characterization of rings over which every R-module is X -Gorenstein injective. These rings are particular cases of the wellknown quasi-Frobenius rings.

Proposition: 2.4 Every R-module is X -Gorenstein injective if and only if every R-module in X is projective. In particular, if the above equivalence conditions are satisfied, then R is quasiFrobenius.

Proof: First, from [3, Theorem 2.2] and its proof, if one of the equivalence conditions are satisfied, then R is quasi-Frobenius.

Now, assume that every R-module is X -Gorenstein injective. Then, from Proposition 2.2, $\mathrm{Ext}^i_R(F, M) = 0$ for Every R-module M, every $F \in X$, and every $i > 0$. Then, every F in X is projective.

Conversely, consider an R-module M . Let $\cdots \to P_1 \to P_0 \to M \to 0$ and $0 \to M \to I_0 \to I_1 \to \cdots$ be injective and projective resolutions of M.

Since, by the reason above, R is quasi Frobenius, every projective R-module is injective. Then, the above projective resolution is a right injective resolution of M . Now, assembling the two above resolutions, we get the following exact sequence: $\cdots \to P_1 \to P_0 \to I_0 \to I_1 \to \cdots$ Since, by hypothesis, every R-module in X is injective, the above exact sequence is clearly an X -complete injective resolution, as desired.

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