

X -GORENSTEIN INJECTIVE MODULES

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ABSTRACT

In this paper, we generalize the notion of Gorenstein injective modules. Namely, we introduce X -Gorenstein injective modules, where X is a class of modules that contains all injective modules. We show that the principal results on Gorenstein injective module remain true for the X -Gorenstein injective modules.

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1. INTRODUCTION:

Throughout this paper, R denotes a non-trivial associative ring with identity, and all modules are left R -modules.

In 1967-69, Auslander and Bridger [1, 2] introduced the G -dimension for finitely generated R -modules when R is Noetherian, denoted by $G - \dim(M)$ where M is a finitely generated R -module. As the classical case, the G -dimension of modules is defined in terms of resolutions by modules of G -dimension 0, which are defined as follows:

A finitely generated R -module M has G -dimension 0, if:

- $\text{Ext}^m(M, R) = 0 = \text{Ext}^m(\text{Hom}_R(M, R), R)$ for every $m > 0$; and RR
- M is reflexive, that is, the canonical map $M \rightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)$ is an isomorphism.

In [1], Auslander proved that a finitely generated R -module M has G -dimension 0 if and only if there exists an exact sequence of finitely generated free R -modules $\mathbf{L} = \cdots \rightarrow L_1 \rightarrow L_0 \rightarrow L^0 \rightarrow L^1 \rightarrow \cdots$ such that $M \sim \text{Im}(L_0 \rightarrow L^0)$ and the complex $\text{Hom}_R(\mathbf{L}, R)$ is exact.

In [5, 6], Enochs and Jenda defined, over arbitrary rings, the Gorenstein injective modules as follows:

Definition: 1.1 An R -module M is said to be Gorenstein injective, if there exists an exact sequence of projective R -modules

$$\mathbf{I} = \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

such that $M \sim \text{Im}(I_0 \rightarrow I^0)$ and such that $\text{Hom}_R(I, -)$ leaves the sequence \mathbf{I} exact whenever I is a injective R -module.

The exact sequence \mathbf{I} is called a complete injective resolution.

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Namely, we define X -Gorenstein injective modules, where X is a class of R-modules that contains all injective R-modules (see Definition 2.1). In Proposition 2.2 we characterize the X -Gorenstein injective modules. Our main result is Theorem 2.3, in which, we study the behavior of the notion of X -Gorenstein injective modules in short exact sequences.

2. X -GORENSTEIN INJECTIVE MODULES:

In this section we investigate the following generalization of Gorenstein injective modules.

Definition: 2.1 Let X be a class of R-modules that contains all injective R-modules. An R-module M is called X -Gorenstein injective, if there exists an exact sequence of injective R-modules $I = \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ such that $M \sim \text{Im}(I_0 \rightarrow I^0)$ and $\text{Hom}_R(F, I)$ is exact whenever $F \in X$.

The sequence I is called an X -complete injective resolution.

We start with the following characterization of an X -Gorenstein injective module.

Proposition 2.2 For an R-module M , the following conditions are equivalent:

1. M is X -Gorenstein injective.
2. (i) $\text{Ext}^i(F, M) = 0$ for every $F \in X$ and every $i > 0$;
 (ii) There exists an exact sequence of R-modules $I = \cdots \rightarrow I_0 \rightarrow I^1 \rightarrow M \rightarrow 0$, where each I_i is injective, such that $\text{Hom}_R(F, I)$ is exact for every $F \in X$.
3. There exists a short exact sequence of R-modules $0 \rightarrow N \rightarrow I \rightarrow M \rightarrow 0$, where I is injective and N is X -Gorenstein injective.
4. There exists a family of short exact sequences of R-modules $0 \rightarrow M_{i+1} \rightarrow I_i \rightarrow M_i \rightarrow 0$ ($i \in \mathbb{Z}$), where each I_i is injective and $M_0 = M$, such that
 $\text{Ext}^1(F, M_i) = 0$ for every $F \in X$ and every $i \in \mathbb{Z}$.

Proof: The proof of the equivalences (1) \Leftrightarrow (2) \Leftrightarrow (4) is analogous to the ones of the Gorenstein injective counterpart (see [4, 7]).

The implication (3) \Rightarrow (4) is obvious.

To end, we prove the implication (3) \Rightarrow (2). Let $F \in X$. Applying the functor

$\text{Hom}_R(F, -)$ to the exact sequence $0 \rightarrow N \rightarrow I \rightarrow M \rightarrow 0$, we get the long exact sequence: $\cdots \rightarrow \text{Ext}_i(F, M) \rightarrow \text{Ext}_i(F, I) \rightarrow \text{Ext}_i(F, N) \rightarrow \cdots$. For every $i > 0$, we have: $\text{Ext}_R(F, N) = 0$ (since N is X -Gorenstein injective and by the equivalent (1) \Leftrightarrow (2)). Also, we have $\text{Ext}_i(F, I) = 0$ (since I is injective).

Then, $\text{Ext}_R(F, M) = 0$ for every $i > 0$.

It remains to prove (ii). Since N is X -Gorenstein injective and by the equivalent (1) \Leftrightarrow (2), there exists an exact sequence of R-modules $I = \cdots \rightarrow I_0 \rightarrow I_1 \rightarrow N \rightarrow 0$, where each I_i is injective, such that $\text{Hom}_R(F, I)$ is exact for all R-modules $F \in X$. Assembling this sequence with the short exact sequence $0 \rightarrow N \rightarrow I \rightarrow M \rightarrow 0$ we get the following exact sequence $E = \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I \rightarrow M \rightarrow 0$ such that the sequence $\text{Hom}_R(F, E)$ is exact for every R-module $F \in X$, as desired.

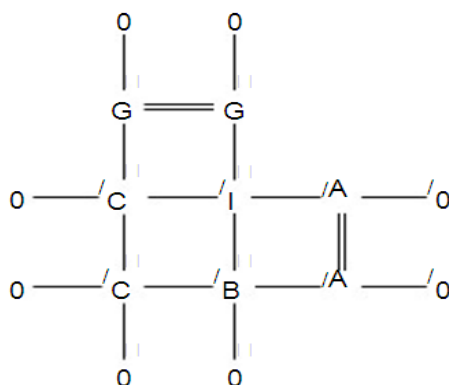
The following result, which investigates the behavior of X -Gorenstein injective modules in short exact sequences, generalizes [7, Theorem 2.6].

Theorem: 2.3

- (1) Let $0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0$ be a short exact sequences of R-modules, where C is X -Gorenstein injective. Then, A is X -Gorenstein injective if and only if B is X -Gorenstein injective.
- (2) Let $(M_i)_{i \in I}$ be a family of R-modules. Then, M_i is X -Gorenstein injective if and only if M_i is X -Gorenstein injective for every $i \in I$.

Proof: The equivalences of both (1) and (2) can be proved similarly to the one of [7, Theorem 2.6]. Here, we give a new and simple proof of the “only if” part of (1). Then, assume that *B* is *X*-Gorenstein injective. By Proposition 2.2 (1) \Leftrightarrow (3), there exists an exact sequence of *R*-modules $0 \rightarrow G \rightarrow I \rightarrow B \rightarrow 0$, where *I* is injective and *G* is *X*-Gorenstein injective. Consider the following

Pullbacks diagram:



Applying the “if” part to the left vertical short exact sequence, we get that *C* is *X*-Gorenstein injective. Therefore, use the equivalence (1) \Leftrightarrow (3) of Proposition 2.2 and the middle horizontal short exact sequence to get that *A* is *X*-Gorenstein injective.

We end the paper with a characterization of rings over which every *R*-module is *X*-Gorenstein injective. These rings are particular cases of the wellknown quasi-Frobenius rings.

Proposition: 2.4 Every *R*-module is *X*-Gorenstein injective if and only if every *R*-module in *X* is projective. In particular, if the above equivalence conditions are satisfied, then *R* is quasiFrobenius.

Proof: First, from [3, Theorem 2.2] and its proof, if one of the equivalence conditions are satisfied, then *R* is quasi-Frobenius.

Now, assume that every *R*-module is *X*-Gorenstein injective. Then, from Proposition 2.2, $\text{Ext}_R^i(F, M) = 0$ for

Every *R*-module *M*, every *F* \in *X*, and every *i* > 0. Then, every *F* in *X* is projective.

Conversely, consider an *R*-module *M*. Let $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ and $0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$ be injective and projective resolutions of *M*.

Since, by the reason above, *R* is quasi Frobenius, every projective *R*-module is injective. Then, the above projective resolution is a right injective resolution of *M*. Now, assembling the two above resolutions, we get the following exact sequence: $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$. Since, by hypothesis, every *R*-module in *X* is injective, the above exact sequence is clearly an *X*-complete injective resolution, as desired.

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