International Research Journal of Pure Algebra -3(9), 2013, 290-295
Available online through www.rjpa.info ISSN 2248–9037

The Almost Generalized Nörlund summability of Conjugate Fourier series

Sandeep Kumar Tiwari*, Uttam Upadhyay and Dalendra Kumar Bhatt*****

****School of studies in Mathematics, Vikram University, Ujjain- (M.P.), India.***

(Received on: 10-08-13; Revised & Accepted on: 30-09-13)

ABSTRACT

In this paper, we obtained the degree of approximation of a functions belonging to the Lip α class by almost generalized Nörlund means of conjugate Fourier series.

Keywords: *Degree of approximation, Lip α class of function, Nörlund mean, Fourier series, Conjugate Fourier series, Lebesgue integral.*

2010 Mathematics Subject Classification: *42B05, 42B08.*

1. DEFINITION AND NOTATION

Let f (x) be a 2π- periodic function and integrable in the Lebesgue sense. The Fourier series f (x) is given by

$$f\left(x\right)\sim\frac{1}{2}a_o+\sum_{n=1}^{\infty}\left(a_n\ \cos\ nx+b_n\sin nx\right)\equiv\sum_{n=0}^{\infty}A_n(x) \tag{1.1}$$

The Conjugate series of Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}\left(a_n\sin nx-b_n\cos nx\right)\equiv\sum_{n=1}^{\infty}B_n(x) \tag{1.2}$$

The degree of approximation of a function f: $R\rightarrow R$ by a trigonometric polynomial t_n of order n is defined by Zygmund [7]

$$\|\,t_n-f\,\|_\infty=Sup\left\{\,|\,t_n(x)-f(x)\,|:x\in R\right\} \tag{1.3}$$

A function $f\in Lip\ \alpha$ if

$$f(x+t)-f(x)=O\left(\,|\,t\,|^{\alpha}\right),\ fo\ \ 0<\alpha\leq 1 \tag{1.4}$$

Lorentz [3] has defined:

Let $\sum_{n=1}^{\infty}a_n$ be an infinite series whose n^{th} partial sum is denoted by S_n. Then the sequence $\{s_n\}$ is said to be almost convergent to a limit s, if

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=p}^{n+p}s_k\rightarrow s \tag{1.5}$$

uniformly with respect to p.

****Corresponding author: Uttam Upadhyay*****

Let $\{p_n\}$ and $\{q_n\}$ be the sequence of positive constants such that

$$P_n = \sum_{k=0}^{n} p_k;$$

$$Q_n = \sum_{k=0}^{n} q_k;$$

and $R_n = \sum_{k=0}^{n} p_k\, q_{n-k} \neq 0\ (n \geq 0)$

where P_n, Q_n and $R_n \to \infty$ as $n \to \infty$.

The series $\sum_{n=1}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be almost generalized Nörlund (N, p_n, q_n) Qureshi [4] summable to s, if

$$t_{n,p} = \frac{1}{R_n} \sum_{k=0}^{n} p_k\, q_{n-k}\, s_{k,p} \tag{1.6}$$

tends to s, as $n\to\infty$,uniformly with respect to p, where

$$s_{k,p} = \frac{1}{k+1} \sum_{r=p}^{k+p} s_r \tag{1.7}$$

We shall use the following notations:

$$(i)\quad \psi(t) = f(x+t) - f(x-t)$$

$$(ii)\quad \overline{K}_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} \frac{p_k q_{n-k}}{k+1} \frac{\cos(k+2p+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}$$

$$(iii)\quad \overline{f}(x) = -\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\left(\frac{t}{2}\right) dt$$

2. MAIN THEOREM

The degree of approximation of functions belonging to Lipschitz class has been discussed by a number of researchers like Chandra[1], Holland[2], Qureshi[4][5]etc. Therefore, the purpose of present paper is to establish a new theorem on the degree of approximation of the function belonging to Lip α class by almost generalized Nörlund means of its conjugate Fourier series. We prove the following:

Theorem 2.1: If $f: R\to R$ is 2π- periodic and Lebesgue integrable on [-π,π] and $f \in$ Lip α class then the degree of approximation of function f by almost generalized Nörlund means of its conjugate Fourier series of f satisfies, for n=0,1,2,3,…

$$\| \overline{t}_{n,p}(x) - \overline{f}(x) \|_\infty = \begin{cases} O\left(\dfrac{1}{(n+1)^\alpha}\right), & 0<\alpha<1 \\ O\left(\dfrac{\log(n+1)\pi e}{(n+1)}\right), & \alpha=1 \end{cases}$$

where $\{p_n\}$ and $\{q_n\}$ are non-negative, monotonic and non-increasing sequence of real constants such that

$$\sum_{k=0}^{n} p_k\, q_{n-k} = O(R_n). \tag{2.1}$$

3. LEMMAS: For the proof of our theorem, we require following lemmas:

Lemma 3.1: $\left|\overline{K}_n(t)\right| = O\left(\frac{1}{t}\right);\quad for\ o\ \le\ t \le \frac{1}{n+1}.$

Proof: $\left|\overline{K}_n(t)\right| \;=\; \left| \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{k+1}\frac{\cos(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \right|$

$$\left|\overline{K}_n(t)\right| \le \frac{1}{2\pi R_n}\left| \sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{k+1}\frac{\cos(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \right|$$

By using for $0 \le t \le \frac{1}{n+1}, \sin nt \le n\sin t, \sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}; we\,have$

$$\begin{aligned}
&\le \frac{1}{2\pi R_n}\left| \sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{k+1}\frac{(k+1)}{t/\pi} \right| \\
&= O\left(\frac{1}{t}\right)\left\{ \frac{1}{R_n}\left| \sum_{k=0}^{n} p_k\, q_{n-k} \right| \right\} \\
&= O\left(\frac{1}{t}\right);\quad by \quad (2.1)
\end{aligned}$$

$$\left|\overline{K}_n(t)\right| = O\left(\frac{1}{t}\right)$$

This complete proof of the lemma (3.1).

Lemma 3.2: $\left|\overline{K}_n(t)\right| = O\left(\frac{1}{(n+1)t^2}\right)\ ;\ for\ \frac{1}{n+1} \le t \le \pi.$

Proof: For $\frac{1}{n+1} \le t \le \pi, and\, \sin\left(\frac{t}{2}\right) \ge \left(\frac{t}{\pi}\right);\quad we\,have$

$$\left|\overline{K}_n(t)\right| = \left| \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{k+1}\frac{\cos(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \right|$$

$$\leq \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n}\frac{p_k\, q_{n-k}}{(k+1)}\right|\left|\frac{1}{t^2/\pi^2}\right|$$

$$= O\left(\frac{1}{(n+1)t^2}\right)\left\{\frac{1}{R_n}\left|\sum_{k=0}^{n} p_k\, q_{n-k}\right|\right\}$$

$$= O\left(\frac{1}{(n+1)t^2}\right); \quad by \quad (2.1)$$

$$\left|\bar{K}_n(t)\right| = O\left(\frac{1}{(n+1)t^2}\right).$$

This complete proof of the lemma (3.2).

4. PROOF OF THE MAIN THEOREM

Let $\bar{s}_n(f;x)$ be the n[th] partial sum of conjugate series (1.2), we have

$$\bar{s}_n(f;x) - \bar{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$$

$$\bar{s}_{k,p}(x) - \bar{f}(x) = \frac{1}{k+1}\sum_{r=p}^{k+p}\left\{\bar{s}_r(x) - \bar{f}(x)\right\}$$

$$= \frac{1}{2\pi(k+1)}\int_0^{\pi}\psi(t)\sum_{r=p}^{k+p}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$$

$$= \frac{1}{2\pi(k+1)}\int_0^{\pi}\psi(t)\sum_{r=p}^{k+p}\frac{\sin pt - \sin(k+p+1)t}{2\sin^2\frac{t}{2}}\,dt$$

Then the almost generalized Nörlund transform of $\bar{s}_n(f;x)$ is given by

$$\bar{t}_{n,p}(x) - \bar{f}(x) = \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\left\{\bar{s}_{k,p}(x) - \bar{f}(x)\right\}$$

$$= \frac{1}{2\pi R_n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{p_k\, q_{n-k}}{k+1}\,\frac{\sin pt - \sin(k+p+1)t}{2\sin^2\frac{t}{2}}\,dt$$

$$= \frac{1}{2\pi R_n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{p_k q_{n-k}}{k+1}\,\frac{\cos(k+2p+1)\frac{t}{2}\,\sin(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

$$= \int_0^{\pi}\psi(t)\,\bar{K}_n(t)\,dt$$

$$= \int_0^{\frac{1}{n+1}} \psi(t)\,\bar{K}_n(t)\,dt + \int_{\frac{1}{n+1}}^{\pi} \psi(t)\,\bar{K}_n(t)\,dt$$

$$= \mathrm{I}_1 + \mathrm{I}_{2,} \quad \text{(Say)} \tag{4.1}$$

Now consider,

$$|I_1| = \left| \int_0^{\frac{1}{n+1}} \psi(t)\,\bar{K}_n(t)\,dt \right|$$

$$\le \int_0^{\frac{1}{n+1}} |\psi(t)|\,|\bar{K}_n(t)|\,dt$$

$$= \int_0^{\frac{1}{n+1}} O(t^\alpha)\,O\left(\frac{1}{t}\right)dt \text{ by lemma (3.1) and } \psi(\mathrm{t}) \in \mathrm{Lip}\ \alpha$$

$$= O\left(\frac{1}{(n+1)^\alpha}\right); \quad 0 < \alpha \le 1 \tag{4.2}$$

Now we consider

$$|I_2| = \left| \int_{\frac{1}{n+1}}^{\pi} \psi(t)\,\bar{K}_n(t)\,dt \right|$$

$$\le \int_{\frac{1}{n+1}}^{\pi} |\psi(t)|\,|\bar{K}_n(t)|\,dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} O(t^\alpha)\,O\left(\frac{1}{(n+1)t^2}\right)dt \text{ by lemma (3.2) and } \psi(\mathrm{t}) \in \mathrm{Lip}\ \alpha$$

$$= O\left(\frac{1}{(n+1)}\right)\left[\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2}\,dt\right]$$

$$= O\left(\frac{1}{(n+1)}\right)\begin{cases} \left(\dfrac{t^{\alpha-1}}{\alpha-1}\right)_{\frac{1}{n+1};}^{\pi} & \alpha \ne 1 \\ (\log t)_{\frac{1}{n+1};}^{\pi} & \alpha = 1 \end{cases}$$

$$= O\left(\frac{1}{(n+1)}\right)\begin{cases} \left(\dfrac{1}{1-\alpha}\right)\left(\dfrac{1}{(n+1)^{\alpha-1}} - \dfrac{1}{\pi^{1-\alpha}}\right); & \alpha \ne 1 \\ \log \pi + \log(n+1); & \alpha = 1 \end{cases}$$

$$= O\left(\frac{1}{(n+1)}\right)\begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha-1}}\right); & 0 < \alpha < 1 \\ O\left(\dfrac{\log\,(n+1)\pi}{(n+1)}\right); & \alpha = 1 \end{cases}$$

$$\left|I_2\right| = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right); & 0<\alpha<1 \\ O\left(\dfrac{\log\,(n+1)\pi}{(n+1)}\right); & \alpha=1 \end{cases} \tag{4.3}$$

Combining (4.1),(4.2) and (4.3) we have

$$\left|\overline{t}_{n,p}(x) - \overline{f}(x)\right| = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right); & 0<\alpha<1 \\ O\left(\dfrac{\log\,(n+1)\pi\, e}{(n+1)}\right); & \alpha=1 \end{cases}$$

Thus,

$$\left\| \overline{t}_{n,p}(x) - \overline{f}(x) \right\|_{\infty} = \underset{-\pi\le x\le\pi}{Sup} \left\{ \left| \overline{t}_{n,p}(x) - \overline{f}(x) \right| :\ x\in R \right\}$$

$$\left\| \overline{t}_{n,p}(x) - \overline{f}(x) \right\|_{\infty} = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right); & 0<\alpha<1 \\ O\left(\dfrac{\log\,(n+1)\pi\, e}{(n+1)}\right); & \alpha=1 \end{cases}$$

This complets the proof of the theorem.

REFERENCES

1. Chandra, P.: On degree of approximation of functions belonging to Lipschitz class, Nanta Math., 8(1975),88.
2. Holland, A.S.B.: A survey of degree of approximation of continouous functions, SIAM Review 23 (3) (1981), 344-379.
3. Lorentz, G.G.: A contribution to the theory of divergent series, Acta. Math., 80 (1948), 167-190.
4. Qureshi, K.: On the degree of approximation of a periodic function *f* by almost Nörlund means, Tamkang J. Math.12(1)(1981), 35-38.
5. Qureshi, K.: On the degree of approximation of a periodic function *f* by almost Riesz means of its conjugate series, Indian J. Pure appl.Math.13 (10) (1982), 1136-1139.
6. Tripathi, L.M.: On almost Nörlund summability of conjugate series of Fourier series, Vijnana Parishad Anushandhan Patrika, 27(1984), 2.
7. Zygmund, A.: Trigonometric series, 2nd Rev. Ed. Cambridge, Uni. Press. cambridge, (1968).

Source of Support: Nil, Conflict of interest: None Declared