



The Almost Generalized Nörlund summability of Conjugate Fourier series

Sandeep Kumar Tiwari*, Uttam Upadhyay** and Dalendra Kumar Bhatt***

*School of studies in Mathematics, Vikram University, Ujjain- (M.P.), India.

(Received on: 10-08-13; Revised & Accepted on: 30-09-13)

ABSTRACT

In this paper, we obtained the degree of approximation of a functions belonging to the Lip α class by almost generalized Nörlund means of conjugate Fourier series.

Keywords: Degree of approximation, Lip α class of function, Nörlund mean, Fourier series, Conjugate Fourier series, Lebesgue integral.

2010 Mathematics Subject Classification: 42B05, 42B08.

1. DEFINITION AND NOTATION

Let $f(x)$ be a 2π - periodic function and integrable in the Lebesgue sense. The Fourier series $f(x)$ is given by

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.1}$$

The Conjugate series of Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{1.2}$$

The degree of approximation of a function $f: R \rightarrow R$ by a trigonometric polynomial t_n of order n is defined by Zygmund [7]

$$\|t_n - f\|_{\infty} = \text{Sup} \{ |t_n(x) - f(x)| : x \in R \} \tag{1.3}$$

A function $f \in \text{Lip } \alpha$ if

$$f(x+t) - f(x) = O(|t|^{\alpha}), \text{ for } 0 < \alpha \leq 1 \tag{1.4}$$

Lorentz [3] has defined:

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series whose n^{th} partial sum is denoted by S_n . Then the sequence $\{s_n\}$ is said to be almost convergent to a limit s , if

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=p}^{n+p} s_k \rightarrow s \tag{1.5}$$

uniformly with respect to p .

Corresponding author: Uttam Upadhyay*

Let $\{p_n\}$ and $\{q_n\}$ be the sequence of positive constants such that

$$P_n = \sum_{k=0}^n p_k;$$

$$Q_n = \sum_{k=0}^n q_k;$$

and $R_n = \sum_{k=0}^n p_k q_{n-k} \neq 0 \ (n \geq 0)$

where P_n, Q_n and $R_n \rightarrow \infty$ as $n \rightarrow \infty$.

The series $\sum_{n=1}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be almost generalized Nörlund (N, p_n, q_n) Qureshi [4] summable to s , if

$$t_{n,p} = \frac{1}{R_n} \sum_{k=0}^n p_k q_{n-k} s_{k,p} \tag{1.6}$$

tends to s , as $n \rightarrow \infty$, uniformly with respect to p , where

$$s_{k,p} = \frac{1}{k+1} \sum_{r=p}^{k+p} s_r \tag{1.7}$$

We shall use the following notations:

(i) $\psi(t) = f(x+t) - f(x-t)$

(ii)
$$\bar{K}_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^n \frac{p_k q_{n-k}}{k+1} \frac{\cos(k+2p+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}$$

(iii)
$$\bar{f}(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \cot\left(\frac{t}{2}\right) dt$$

2. MAIN THEOREM

The degree of approximation of functions belonging to Lipschitz class has been discussed by a number of researchers like Chandra[1], Holland[2], Qureshi[4][5]etc. Therefore, the purpose of present paper is to establish a new theorem on the degree of approximation of the function belonging to Lip α class by almost generalized Nörlund means of its conjugate Fourier series. We prove the following:

Theorem 2.1: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is 2π - periodic and Lebesgue integrable on $[-\pi, \pi]$ and $f \in \text{Lip } \alpha$ class then the degree of approximation of function f by almost generalized Nörlund means of its conjugate Fourier series of f satisfies, for $n=0,1,2,3,\dots$

$$\| \bar{t}_{n,p}(x) - \bar{f}(x) \|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right), & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi e}{(n+1)}\right), & \alpha = 1 \end{cases}$$

where $\{p_n\}$ and $\{q_n\}$ are non-negative, monotonic and non-increasing sequence of real constants such that

$$\sum_{k=0}^n p_k q_{n-k} = O(R_n). \tag{2.1}$$

3. LEMMAS: For the proof of our theorem, we require following lemmas:

Lemma 3.1: $|\bar{K}_n(t)| = O\left(\frac{1}{t}\right)$; for $0 \leq t \leq \frac{1}{n+1}$.

Proof: $|\bar{K}_n(t)| = \left| \frac{1}{2\pi R_n} \sum_{k=0}^n \frac{p_k q_{n-k}}{k+1} \frac{\cos(k+2p+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}} \right|$

$$|\bar{K}_n(t)| \leq \frac{1}{2\pi R_n} \left| \sum_{k=0}^n \frac{p_k q_{n-k}}{k+1} \frac{\cos(k+2p+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}} \right|$$

By using for $0 \leq t \leq \frac{1}{n+1}$, $\sin nt \leq n \sin t$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$; we have

$$\begin{aligned} &\leq \frac{1}{2\pi R_n} \left| \sum_{k=0}^n \frac{p_k q_{n-k}}{k+1} \frac{(k+1)}{t/\pi} \right| \\ &= O\left(\frac{1}{t}\right) \left\{ \frac{1}{R_n} \left| \sum_{k=0}^n p_k q_{n-k} \right| \right\} \\ &= O\left(\frac{1}{t}\right); \text{ by (2.1)} \end{aligned}$$

$$|\bar{K}_n(t)| = O\left(\frac{1}{t}\right)$$

This complete proof of the lemma (3.1).

Lemma 3.2: $|\bar{K}_n(t)| = O\left(\frac{1}{(n+1)t^2}\right)$; for $\frac{1}{n+1} \leq t \leq \pi$.

Proof: For $\frac{1}{n+1} \leq t \leq \pi$, and $\sin\left(\frac{t}{2}\right) \geq \left(\frac{t}{\pi}\right)$; we have

$$|\bar{K}_n(t)| = \left| \frac{1}{2\pi R_n} \sum_{k=0}^n \frac{p_k q_{n-k}}{k+1} \frac{\cos(k+2p+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}} \right|$$

$$\begin{aligned} &\leq \frac{1}{2\pi R_n} \left| \sum_{k=0}^n \frac{P_k q_{n-k}}{(k+1)} \right| \left| \frac{1}{t^2/\pi^2} \right| \\ &= O\left(\frac{1}{(n+1)t^2}\right) \left\{ \frac{1}{R_n} \left| \sum_{k=0}^n P_k q_{n-k} \right| \right\} \\ &= O\left(\frac{1}{(n+1)t^2}\right); \text{ by (2.1)} \end{aligned}$$

$$|\bar{K}_n(t)| = O\left(\frac{1}{(n+1)t^2}\right).$$

This complete proof of the lemma (3.2).

4. PROOF OF THE MAIN THEOREM

Let $\bar{s}_n(f;x)$ be the n^{th} partial sum of conjugate series (1.2), we have

$$\bar{s}_n(f;x) - \bar{f}(x) = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos\left(n + \frac{1}{2}\right)t}{\sin \frac{t}{2}} dt$$

$$\begin{aligned} \bar{s}_{k,p}(x) - \bar{f}(x) &= \frac{1}{k+1} \sum_{r=p}^{k+p} \{ \bar{s}_r(x) - \bar{f}(x) \} \\ &= \frac{1}{2\pi(k+1)} \int_0^\pi \psi(t) \sum_{r=p}^{k+p} \frac{\cos\left(r + \frac{1}{2}\right)t}{\sin \frac{t}{2}} dt \\ &= \frac{1}{2\pi(k+1)} \int_0^\pi \psi(t) \sum_{r=p}^{k+p} \frac{\sin pt - \sin(k+p+1)t}{2\sin^2 \frac{t}{2}} dt \end{aligned}$$

Then the almost generalized Nörlund transform of $\bar{s}_n(f;x)$ is given by

$$\begin{aligned} \bar{t}_{n,p}(x) - \bar{f}(x) &= \frac{1}{R_n} \sum_{k=0}^n p_k q_{n-k} \{ \bar{s}_{k,p}(x) - \bar{f}(x) \} \\ &= \frac{1}{2\pi R_n} \int_0^\pi \psi(t) \sum_{k=0}^n \frac{p_k q_{n-k}}{k+1} \frac{\sin pt - \sin(k+p+1)t}{2\sin^2 \frac{t}{2}} dt \\ &= \frac{1}{2\pi R_n} \int_0^\pi \psi(t) \sum_{k=0}^n \frac{p_k q_{n-k}}{k+1} \frac{\cos(k+2p+1)\frac{t}{2} \sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}} dt \\ &= \int_0^\pi \psi(t) \bar{K}_n(t) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\frac{1}{n+1}} \psi(t) \bar{K}_n(t) dt + \int_{\frac{1}{n+1}}^{\pi} \psi(t) \bar{K}_n(t) dt \\
 &= I_1 + I_2, \text{ (Say)}
 \end{aligned}
 \tag{4.1}$$

Now consider,

$$\begin{aligned}
 |I_1| &= \left| \int_0^{\frac{1}{n+1}} \psi(t) \bar{K}_n(t) dt \right| \\
 &\leq \int_0^{\frac{1}{n+1}} |\psi(t)| |\bar{K}_n(t)| dt \\
 &= \int_0^{\frac{1}{n+1}} O(t^\alpha) O\left(\frac{1}{t}\right) dt \text{ by lemma (3.1) and } \psi(t) \in \text{Lip } \alpha \\
 &= O\left(\frac{1}{(n+1)^\alpha}\right); \quad 0 < \alpha \leq 1
 \end{aligned}
 \tag{4.2}$$

Now we consider

$$\begin{aligned}
 |I_2| &= \left| \int_{\frac{1}{n+1}}^{\pi} \psi(t) \bar{K}_n(t) dt \right| \\
 &\leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)| |\bar{K}_n(t)| dt \\
 &= \int_{\frac{1}{n+1}}^{\pi} O(t^\alpha) O\left(\frac{1}{(n+1)t^2}\right) dt \text{ by lemma (3.2) and } \psi(t) \in \text{Lip } \alpha \\
 &= O\left(\frac{1}{(n+1)}\right) \left[\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2} dt \right] \\
 &= O\left(\frac{1}{(n+1)}\right) \begin{cases} \left(\frac{t^{\alpha-1}}{\alpha-1}\right)_{\frac{1}{n+1}}^{\pi}; & \alpha \neq 1 \\ (\log t)_{\frac{1}{n+1}}^{\pi}; & \alpha = 1 \end{cases} \\
 &= O\left(\frac{1}{(n+1)}\right) \begin{cases} \left(\frac{1}{1-\alpha}\right) \left(\frac{1}{(n+1)^{\alpha-1}} - \frac{1}{\pi^{1-\alpha}}\right); & \alpha \neq 1 \\ \log \pi + \log(n+1); & \alpha = 1 \end{cases} \\
 &= O\left(\frac{1}{(n+1)}\right) \begin{cases} O\left(\frac{1}{(n+1)^{\alpha-1}}\right); & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi}{(n+1)}\right); & \alpha = 1 \end{cases}
 \end{aligned}$$

$$|I_2| = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right); & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi}{(n+1)}\right); & \alpha = 1 \end{cases} \quad (4.3)$$

Combining (4.1),(4.2) and (4.3) we have

$$|\bar{t}_{n,p}(x) - \bar{f}(x)| = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right); & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi e}{(n+1)}\right); & \alpha = 1 \end{cases}$$

Thus,

$$\|\bar{t}_{n,p}(x) - \bar{f}(x)\|_\infty = \text{Sup}_{-\pi \leq x \leq \pi} \left\{ |\bar{t}_{n,p}(x) - \bar{f}(x)| : x \in R \right\}$$

$$\|\bar{t}_{n,p}(x) - \bar{f}(x)\|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right); & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi e}{(n+1)}\right); & \alpha = 1 \end{cases}$$

This completes the proof of the theorem.

REFERENCES

1. Chandra, P.: On degree of approximation of functions belonging to Lipschitz class, Nanta Math., 8(1975),88.
2. Holland, A.S.B.: A survey of degree of approximation of continuous functions, SIAM Review 23 (3) (1981), 344-379.
3. Lorentz, G.G.: A contribution to the theory of divergent series, Acta. Math., 80 (1948), 167-190.
4. Qureshi, K.: On the degree of approximation of a periodic function f by almost Nörlund means, Tamkang J. Math.12(1)(1981), 35-38.
5. Qureshi, K.: On the degree of approximation of a periodic function f by almost Riesz means of its conjugate series, Indian J. Pure appl.Math.13 (10) (1982), 1136-1139.
6. Tripathi, L.M.: On almost Nörlund summability of conjugate series of Fourier series, Vijnana Parishad Anushandhan Patrika, 27(1984), 2.
7. Zygmund, A.: Trigonometric series, 2nd Rev. Ed. Cambridge, Uni. Press. cambridge, (1968).

Source of Support: Nil, Conflict of interest: None Declared