International Research Journal of Pure Algebra -3(9), 2013, 296-300
Available online through www.rjpa.info ISSN 2248–9037

A COMMON FIXED POINT THEOREM IN METRIC SPACES FOR AN ASYMPTOTICALLY REGULAR SELF MAP

K. P. R. Sastry[1], A. Chandra Sekhar[2] and K. Sujatha[3*]

[1]8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam -530 017, India.

[2]Department of Engineering Mathematics, GIT, Gitam University, Visakhapatnam -530 045, India.

[3]Department of Mathematics, St. Joseph's College for Women, Visakhapatnam-530 004, India.

(Received on: 15-09-13; Revised & Accepted on: 22-09-13)

ABSTRACT

In this paper we prove a fixed point theorem in a metric space, without using continuity. Incidentally we observe that the result of K. Prudhvi [14] is not valid for Cone Metric Spaces. We also observe that it is valid for metric spaces and follows from our result.

Keywords: *Metric Space, Fixed Point, Asymptotically Regular, φ- Contraction.*

Mathematics Subject Classification: *46L06; 39B82; 39B52.*

1 INTRODUCTION

In 2006, P.D. Proinov [13] obtained two types of generalizations of Banach fixed point theorem. The first type involves Meir-Keeler [9] type conditions (see, for instance, Cho *et al*., [3], Lim [8], Park and Rhoades [11]) and the second type involves contractive guage functions (see, for instance, Boyd and Wong [1] and Kim *et al.*, [7]). Proinov [12] obtained equivalence between these two types of contractive conditions and also obtained a new fixed point theorem generalizing some fixed point theorems of Jachymski [6] (see Proinov [12] Theorem 4.1) into multi valued maps. K.Prudhvi [15] proved a Common Fixed Point Theorem for Asymtotically Regular Multivalued Three Maps. Their result generalizes and extends some recent results of S.L. Singh *et al*. [17] for three maps. Also K. Prudhvi [14] proved a fixed point theorm for a continuous self map on a Cone Metric Space. His result generalizes and extends the results Proinov [13]. We observe that the result of Prudhvi [14] is not really a result in Cone Metric Spaces. In this paper however, we prove a metri space version of the result of Prudhvi [14] without using continuity of the self map.

WE BEGIN WITH TWO DEFINITIONS

1.1 Definition ([13], Definition 2.1(i)): Let Φ denote the class of all functions $\varphi: R^+ \to R^+$ such that φ is increasing and for any $\mathcal{E} > 0$, $\exists\ \delta > \mathcal{E} \ni \mathcal{E} \le t < \delta \Longrightarrow \varphi(t) < \mathcal{E}$.

Asymptotic regularity for single- valued maps is due to Brower and Petryshyn [4].

1.2 Definition [4]: A self-map T on a metric space (X, d) is asymptotic regular

if $\exists\ x_0 \in X \ni d(T^n x_0, T^{n+1} x_0) \to 0$ as $n \to \infty$.

K. Prudhvi [14] proved the following fixed point theorem for a continuous self map on a Cone Metric Space. His result generalizes and extends the results of Proinov [13]. For relevant definitions we may refer to [14]. Mappings considered in [14] are called φ – contractions.

**Corresponding author: K. Sujatha[3*]*
[3]Department of Mathematics, St. Joseph's College for Women, Visakhapatnam-530 004, India.
E-mail: kambhampati.sujatha@yahoo.com

1.3 Theorem ([14] K. Prudhvi, Theorem 2.2)

Let T be a continuous and asymptotically regular self-mapping on a complete cone metric space (X, d) and P be an order cone satisfying the following conditions:

$d(Tx, Ty) \leq \varphi(D(x, y))$, for all $x, y \in X$; (1.3.1)

where, $D(x, y) = d(x, y) + \gamma[d(x, Tx) + d(y, Ty)]$, $0 \leq \gamma \leq 1$.

Then T has a unique fixed point.

But we observe that the above result (1.3.1) of Prudhvi [14] is not really a result in Cone Metric Spaces since (1.3.1) is not meaningful, φ being real valued. In this paper however, we prove a metric space version of the result of Prudhvi [14] without using continuity of the self map.

2. MAIN RESULT

In this section we prove the metric space version of Theorem 1.3 without assuming continuity of T.

2.1 Theorem

Let T be an asymptotically regular self-mapping on a complete metric space (X, d) satisfying the following condition:

there exist $\gamma \in [0,1)$ and $\varphi \in \Phi$ such that

$d(Tx, Ty) \leq \varphi(D(x, y))$, for all $x, y \in X$; … (2.1.1)

where $D(x, y) = d(x, y) + \gamma[d(x, Tx) + d(y, Ty)]$.

Then T has a unique fixed point.

Proof: Since T is asymptotically regular,

$\exists\, x_0 \in X$ such that $d(T^n x_0, T^{n+1} x_0) \to 0$ as $n \to \infty$

Write $x_n = T^n x_0$ and $\alpha_n = d(x_n, x_{n+1})$, n = 1, 2,…

so that $\alpha_n \to 0$ as $n \to \infty$

Let $\mathcal{E} > 0$. Since $\varphi \in \Phi$, $\exists\, \delta \ni \mathcal{E} < \delta < 2\mathcal{E}$ such that

$\mathcal{E} \leq t < \delta \Rightarrow \varphi(t) < \mathcal{E}$

Since $\alpha_n \to 0\ \exists\ N \ni \alpha_n < \frac{\delta - \mathcal{E}}{1+2\gamma}\ \forall\, n \geq N$

$\therefore d(x_n, x_{n+1}) < \frac{\delta - \mathcal{E}}{1+2\gamma} < \frac{\delta + 2\mathcal{E}\gamma}{1+2\gamma}\ \forall\, n \geq N$ (2.1.2)

Now we show that $d(x_n, x_{n+k}) < \frac{\delta + 2\mathcal{E}\gamma}{1+2\gamma}$ where $n \geq N$ and k = 1,2,… (2.1.3)

We prove this by induction.

(2.1.3) is true for k = 1 and all $n \geq N$, by (2.1.2)

Suppose $d(x_n, x_{n+k}) < \frac{\delta + 2\mathcal{E}\gamma}{1+2\gamma}$ where $n \geq N$ and $k \geq 1$ (2.1.4)

We prove (2.1.3) for k+1.

We observe that $\varphi(\mathcal{E}) < \mathcal{E}$ ($\because \varphi(\mathcal{E}) \leq \varphi(t) < \mathcal{E}\ \forall\, t \in [\mathcal{E}, \delta)$)

Now $d(x_n, x_{n+k+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k+1})$

$$< \frac{\delta-\varepsilon}{1+2\gamma} + d(x_{n+1}, x_{n+k+1}) \quad \ldots \tag{2.1.5}$$

Now $d(x_{n+1}, x_{n+k+1}) = d(T^{n+1} x_0, T^{n+k+1} x_0)$

$$= d(T(T^n x_0), T(T^{n+k} x_0))$$

$$\leq \varphi(D(T^n x_0, T^{n+k} x_0)) \tag{2.1.6}$$

Now, $D(T^n x_0, T^{n+k} x_0) = d(T^n x_0, T^{n+k} x_0) + \gamma\,[d(T^n x_0, T(T^n x_0)) + d(T^{n+k} x_0), T(T^{n+k} x_0))]$

$$= d(T^n x_0, T^{n+k} x_0) + \gamma(\alpha_n + \alpha_{n+1})$$

$$< d(T^n x_0, T^{n+k} x_0) + \gamma\left[\frac{\delta-\varepsilon}{1+2\gamma} + \frac{\delta-\varepsilon}{1+2\gamma}\right]$$

$$= d(x_n, x_{n+1}) + 2\gamma\left(\frac{\delta-\varepsilon}{1+2\gamma}\right)$$

$$< \frac{\delta+2\varepsilon\gamma}{1+2\gamma} + 2\gamma\left(\frac{\delta-\varepsilon}{1+2\gamma}\right) \text{ (by (2.1.4))}$$

$$= \frac{(1+2\gamma)\delta}{1+2\gamma} = \delta$$

$\therefore D(T^n x_0, T^{n+k} x_0) < \delta$

Case (i): $\varepsilon \leq D(T^n x_0, T^{n+k} x_0)$. Then

$\varphi(D(T^n x_0, T^{n+k} x_0)) < \varepsilon$ ($\because \varphi \in \Phi$)

$\therefore d(x_n, x_{n+k+1}) < \frac{\delta-\varepsilon}{1+2\gamma} + \varepsilon = \frac{\delta+2\varepsilon\gamma}{1+2\gamma}$ (from (2.1.5))

Case (ii): $\varepsilon > D(T^n x_0, T^{n+k} x_0)$. Then

$d(x_n, x_{n+k+1}) < \frac{\delta-\varepsilon}{1+2\gamma} + d(x_{n+1}, x_{n+k+1})$

$$< \frac{\delta-\varepsilon}{1+2\gamma} + \varphi(D(T^n x_0, T^{n+k} x_0)) \text{ (from (2.1.6))}$$

$$\leq \frac{\delta-\varepsilon}{1+2\gamma} + \varphi(\varepsilon) \quad (\because \varphi \text{ is increasing})$$

$$< \frac{\delta-\varepsilon}{1+2\gamma} + \varepsilon = \frac{\delta+2\varepsilon\gamma}{1+2\gamma}$$

$d(x_n, x_{n+k+1}) < \frac{\delta+2\varepsilon\gamma}{1+2\gamma}$

$\therefore$ by induction, $d(x_n, x_{n+k}) < \frac{\delta+2\varepsilon\gamma}{1+2\gamma}$, $n \geq N$ and $k = 1, 2, \ldots$

$\therefore \{x_n\}$ is a Cauchy sequence in X.

Since X is a complete metric space, $\{x_n\}$ converges to a point $x \in X$

Now $d(Tx, x_{n+1}) = d(Tx, Tx_n)$

$$\leq \varphi(D(x, x_n))$$

$D(x, x_n) = d(x, x_n) + \gamma[d(x, Tx) + d(x_n, T x_n)]$

$= d(x, x_n) + \gamma[d(x, Tx) + d(x_n, x_{n+1})]$

$\to \gamma\, d(x, T x)$ as $n \to \infty$

$\therefore$ $D(x, x_n) < \gamma\, d(x, Tx) + \eta$ where $\eta > 0$, for large n

$\therefore$ $d(Tx, x_{n+1}) \leq \varphi(\gamma\, d(x, Tx) + \eta)$

On letting $n \to \infty$, $d(x, Tx) \leq \varphi(\gamma d(x, Tx) + \eta)$

$< \gamma\, d(x, Tx) + \eta$ for small $\eta > 0$, (since $0 \leq \gamma < 1$)

$\therefore$ $d(x, Tx) = 0$

$\therefore x = Tx$

$\therefore$ x is a fixed point of T

Uniqueness: Let w be another fixed point of T.

Then $d(x, w) = d(Tx, Tw)$

$\leq \varphi(D(x, w))$

$= \varphi[d(x, w) + \gamma(d(x, Tx) + d(w, T w))]$

$= \varphi[d(x, w) + \gamma(d(x, x) + d(w, w))]$

$\leq \varphi[d(x, w)]$

$< d(x, w)$ $(\because \varphi(\varepsilon) < \varepsilon)$

which is a contradiction, if $x \neq w$

$\therefore x = w$

Thus, T has a unique fixed point.

ACKNOWLEDGEMENTS

The third author is grateful to (i) the Principal Dr. Sr. N.D. Veronica and the management of St. Joseph's College for Women(A),Visakhapatnam for providing the necessary permission and (ii) the authorities of GITAM for providing necessary facilities to carry on this research.

REFERENCES

[1] M. Abbas and B.E. Rhoades, Fixed and periodic point results in cone metric Spaces. *Appl. Math. Lett.* 22(2009), 511-515.

[2] V. Berinde, A common fixed point theorem for compatible quasi contractive self mappings in metric spaces, *Appl. Math .Comput,* 213(2009), 348-354.

[3] D.W. Boyd and J.S.W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.*, 20(1969), 458-464.

[4] F.E. Brower and W.V. Petryshyn, The solution by iteration of nonlinear functional equations in banach spaces, *Bull. Amer. Math. Soc.*, 72(1966), 571-575.

[5] Y.J. Cho, P.P. Murthy and G. Jungck, A theorem of Meer-Keeler type revisited, *Int. J. Math. Math. Sci.*, 23(2000), 507-511.

[6] B. Fisher, Four mappings with a common fixed point *J. Univ .Kuwait Sci*., (1981), 131-139.

[7] J. Jachymski, Equivalent conditions and the Meir-Keeler type theorems, *J. Math. Anal. Appal.*, 194(1995), 293-303.

[8] G. Jungck, Common fixed points for non continuous non self maps on non-metric spaces, *Far East J. Math. Sci.* (FJMS) 4 (1996) 199-215.

[9] K.H. Kim, S.M. Kang and Y.J. Cho, Common fixed point of Φ-contactive mappings, *East. Asian Math. J.*, 15(1969), 211-222.

[10] T.C. Lim, On characterization of Meer-Keeler contactive maps, *Nonlinear Anal.*, 46(2001), 113-120.

[11] A. Meir and E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, 28(1969), 326-329.

[12] S. Park and B.E. Rhoades, Meer-Keeler type contactive conditions, *Math. Japon.*, 26(1981), 13-20.

[13] P.D. Proinov, Fixed point theorems in metric spaces, *Nonlinear Anal.*, 64(2006), 546- 557.

[14] K. Prudhvi , A Common Fixed Point Theorem in Cone Metric Spaces, Mathematical Theory and Modeling, Vol.3, No.1, 2013, 48-51.

[15] K. Prudhvi, A Common Fixed Point Theorem for Asymptotically Regular Multi-Valued Three Maps, Gen. Math. Notes, Vol. 16, No. 1, May, 2013, pp. 20-25

[16] S.L. Singh, Apichai Hematulin and R.P. Pant, New coincidence and common fixed point theorem, *Applied General Topology* 10 (2009), no.1, 121-130.

[17] S.L. Singh, S.N. Mishra and R. Pant, New fixed point theorems for asymptotically regular multi-valued maps, *Nonlinear Analysis*, 71(2009), 3299-3304.

Source of Support: Nil, Conflict of interest: None Declared