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FEW INDEFINITE INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTION

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ABSTRACT

In this paper we have developed some indefinite integrals involving Hypergeometric function and inverse Trigonometric function. The results represent here are assume to be new.

Key Words and Phrases: *Hypergeometric function, Elliptic integral.*

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1. INTRODUCTION AND PRELIMINARIES

ELLIPTIC INTEGRAL

In integral calculus, elliptic integrals originally arose in connection with the problem of giving the arc length of an ellipse. They were first studied by Giulio Fagnano and Leonhard Euler. Modern mathematics defines an “elliptic integral” as any function f which can be expressed in the form

$$f(x) = \int_c^x R\left(t, \sqrt{P(t)}\right) dt \tag{1.1}$$

where R is a rational function of its two arguments, P is a polynomial of degree 3 or 4 with no repeated roots, and c is a constant.

In general, elliptic integrals cannot be expressed in terms of elementary functions. Exceptions to this general rule are when P has repeated roots, or when R(x, y) contains no odd powers of y. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e. the elliptic integrals of the first, second and third kind).

Besides the Legendre form , the elliptic integrals may also be expressed in Carlson symmetric form. Additional insight into the theory of the elliptic integral may be gained through the study of the Schwarz-Christoffel mapping. Historically, elliptic functions were discovered as inverse functions of elliptic integrals.

Incomplete elliptic integrals are functions of two arguments; complete elliptic integrals are functions of a single argument.

The incomplete elliptic integral of the first kind F is defined as

$$F(\psi, k) = F(\psi \mid k^2) = F(\sin\psi; k) = \int_0^\psi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{1.2}$$

This is trigonometric form of the integral; substituting t=sin θ, x= sin ψ, one obtains Jacobi’s form:

$$F(x; k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{1.3}$$

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Equivalently, in terms of the amplitude and modular angle, one has:

$$F(\psi\backslash\alpha) = F(\psi, \sin\alpha) = \int_0^{\psi} \frac{d\theta}{\sqrt{1-(\sin\theta\sin\alpha)^2}} \tag{1.4}$$

In this notation, the use of vertical bar as delimiter indicates that the argument following it is the " parameter" (as defined above) , while the backslash indicates that it is the modular angle. The use of semicolon implies that the argument proceding it is the sine of amplitude:

$$F(\psi, \sin\alpha) = F(\psi \mid \sin^2\alpha) = F(\psi\backslash\alpha) = F(\sin\psi; \sin\alpha) \tag{1.5}$$

Incomplete elliptic integral of the second kind E is defined as

$$E(\psi, k) = E(\psi \mid k^2) = E(\sin\psi; k) = \sqrt{1-k^2\sin^2\theta}\ d\theta \tag{1.6}$$

Substituting t=sin θ, x= sin ψ, one obtains Jacobi's form:

$$E(x;k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\ dt \tag{1.7}$$

Equivalently, in terms of the amplitude and modular angle, one has:

$$E(\psi\backslash\alpha) = E(\psi, \sin\alpha) = \int_0^{\psi} \sqrt{1-(\sin\theta\sin\alpha)^2}\ d\theta \tag{1.8}$$

Incomplete elliptic integral of the third kind Π is defined as

$$\Pi(n; \psi\backslash\alpha) = \int_0^{\psi} \frac{1}{1-n\sin^2\theta} \frac{d\theta}{1-(\sin\theta\sin\alpha)^2} \tag{1.9}$$

or

$$\Pi(n; \psi \mid m) = \int_0^{\sin\psi} \frac{1}{1-nt^2} \frac{dt}{(1-mt^2)(1-t^2)} \tag{1.10}$$

The number n is called the characteristic and can take on any value, independently of the other arguments.

Complete elliptic integral of the first kind is defined as

Elliptic Integrals are said to be 'complete' when the amplitude $\psi = \frac{\pi}{2}$ and therefore x=1. The complete elliptic integral of the first kind K may thus be defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{1.11}$$

Or more compactly in terms of the incomplete integral of the first kind as

$$K(k) = F\left(\frac{\pi}{2}, k\right) = F(1;k) \tag{1.12}$$

It can be expressed as a power series

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[P_{2n}(0)\right]^2 k^{2n} \tag{1.13}$$

where Pn is the Legendre polynomial, which is equivalent to

$$K(k)=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k^4+\dots\dots+\left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2k^{2n}+\dots\dots\right] \tag{1.14}$$

where n!! denotes the double factorial. In terms of Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$K(k)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) \tag{1.15}$$

The complete elliptic integral of the first kind is sometimes called the quarter period. It can most efficiently be computed in terms of the arithmetic-geometric mean:

$$K(k)=\frac{\frac{\pi}{2}}{agm(1-k,1+k)} \tag{1.16}$$

Complete elliptic integral of second kind is defined as

The complete elliptic integral of the second kind E is proportional to the circumference of the circle C:

$$C=4aE(e)$$

where α is the semi-major axis, and e is the ecentricity. E may be defined as

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}\;d\theta=\int_0^1\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\;dt \tag{1.17}$$

Or more compactly in terms of the incomplete integral of the second kind as

$$E(k)=E\left(\frac{\pi}{2},k\right)=E(1;k) \tag{1.18}$$

It can be expressed as power series

$$E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2\frac{k^{2n}}{1-2n} \tag{1.19}$$

Which is equivalent to

$$E(k)=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2\frac{k^2}{1}-\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\frac{k^4}{3}-\dots\dots-\left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2\frac{k^{2n}}{2n-1}-\dots\dots\right] \tag{1.20}$$

In terms of the Gauss hypergeometric function, the complete elliptic integral of the second kind can be expressed as

$$E(k)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;k^2\right) \tag{1.21}$$

The complete elliptic integral of third kind is defined as

$$\Pi(n,k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{1.22}$$

Generalized Hypergeometric functions

A generalized hypergeometric function pFq (a_1, a_2,.......a_p; b_1, b_2,..........b_q; z) is a function which can be defined in the form of hypergeometric series, i.e. , a series for which successive terms can be written as

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\ z. \tag{1.23}$$

Where k+1 in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{ccc} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{1.24}$$

Where the parameters b_1, b_2,...,b_q are positive integers.

2. MAIN INTEGRALS

$$\int \tan(a\sin^{-1}z)\ dz = \frac{1}{2(4a^2-1)}\ e^{-2\iota\sin^{-1}z}\times$$
$$\times\Bigg[(2a-1)\Bigg\{(2a+1)e^{\iota\sin^{-1}z}\Bigg(e^{2\iota\sin^{-1}z}{}_2F_1\Bigg(1,\frac{1}{2a};1+\frac{1}{2a};-e^{2\iota a\sin^{-1}z}\Bigg)-$$
$$-{}_2F_1\Bigg(1,-\frac{1}{2a};1-\frac{1}{2a};-e^{2\iota a\sin^{-1}z}\Bigg)\Bigg)-e^{\iota(2a+3)\sin^{-1}z}{}_2F_1\Bigg(1,1+\frac{1}{2a};2+\frac{1}{2a};-e^{2\iota a\sin^{-1}z}\Bigg)\Bigg\}$$
$$-(2a+1)e^{\iota(2a\sin^{-1}z+\sin^{-1}z)}\ {}_2F_1\Bigg(1,1-\frac{1}{2a};2-\frac{1}{2a};-e^{2\iota a\sin^{-1}z}\Bigg)\Bigg]+Constant \tag{2.1}$$

$$\int \cot(a\sin^{-1}z)\ dz = \frac{1}{2(4a^2-1)}\ e^{-2\iota\sin^{-1}z}\times$$
$$\times\Bigg[-(2a+1)e^{\iota(2a\sin^{-1}z+\sin^{-1}z)}\ {}_2F_1\Bigg(1,1-\frac{1}{2a};2-\frac{1}{2a};e^{2\iota a\sin^{-1}z}\Bigg)-$$
$$-(2a-1)\Bigg\{e^{\iota(2a+3)\sin^{-1}z}\ {}_2F_1\Bigg(1,1+\frac{1}{2a};2+\frac{1}{2a};e^{2\iota a\sin^{-1}z}\Bigg)+$$
$$+(2a+1)e^{\iota\sin^{-1}z}\Bigg(e^{2\iota\sin^{-1}z}\ {}_2F_1\Bigg(1,\frac{1}{2a};1+\frac{1}{2a};e^{2\iota a\sin^{-1}z}\Bigg)-$$
$$-{}_2F_1\Bigg(1,-\frac{1}{2a};1-\frac{1}{2a};e^{2\iota a\sin^{-1}z}\Bigg)\Bigg)\Bigg\}\Bigg]+Constant \tag{2.2}$$

$$\int \sec(a\sin^{-1}z)\ dz = -\frac{1}{(a^2-1)}\ \iota\ e^{-\iota\sin^{-1}z}\times$$
$$\times\Bigg[(a+1)e^{\iota a\sin^{-1}z}\ {}_2F_1\Bigg(1,\frac{a-1}{2a};\frac{3}{2}-\frac{1}{2a};-e^{2\iota a\sin^{-1}z}\Bigg)+$$
$$+(a-1)e^{\iota(a+2)\sin^{-1}z}\ {}_2F_1\Bigg(1,\frac{a+1}{2a};\frac{3}{2}+\frac{1}{2a};-e^{2\iota a\sin^{-1}z}\Bigg)\Bigg]+Constant \tag{2.3}$$

$$\int \csc(a\sin^{-1}z)\,dz = -\frac{1}{(a^2-1)}\,e^{\iota(a-1)\sin^{-1}z}\times$$
$$\times\Big[(a+1)\;{}_2F_1\Big(1,\frac{a-1}{2a};\frac{3}{2}-\frac{1}{2a};e^{2\iota a\sin^{-1}z}\Big)+$$
$$+(a-1)e^{2\iota\sin^{-1}z}\;{}_2F_1\Big(1,\frac{a+1}{2a};\frac{3}{2}+\frac{1}{2a};e^{2\iota a\sin^{-1}z}\Big)\Big]+Constant \tag{2.4}$$

$$\int \sec(a\cos^{-1}z)\,dz = \frac{1}{a^2-1}\,e^{\iota(a-1)\cos^{-1}z}\times$$
$$\times\Big[(a-1)\;e^{2\iota\cos^{-1}z}\;{}_2F_1\Big(1,\frac{a+1}{2a};\frac{3}{2}+\frac{1}{2a};-e^{2\iota a\cos^{-1}z}\Big)-$$
$$-(a+1)\;{}_2F_1\Big(1,\frac{a-1}{2a};\frac{3}{2}-\frac{1}{2a};-e^{2\iota a\cos^{-1}z}\Big)\Big]+Constant \tag{2.5}$$

$$\int \csc(a\cos^{-1}z)\,dz = \frac{1}{a^2-1}\;\iota\;e^{-\iota\cos^{-1}z}\times$$
$$\times\Big[(a+1)e^{\iota a\cos^{-1}z}\;{}_2F_1\Big(1,\frac{a-1}{2a};\frac{3}{2}-\frac{1}{2a};e^{2\iota a\cos^{-1}z}\Big)-$$
$$-(a-1)e^{\iota(a+2)\cos^{-1}z}\;{}_2F_1\Big(1,\frac{a+1}{2a};\frac{3}{2}+\frac{1}{2a};e^{2\iota a\cos^{-1}z}\Big)\Big]+Constant \tag{2.6}$$

$$\int \tan(a\cos^{-1}z)\,dz = \frac{1}{2(4a^2-1)}\iota\;e^{-2\iota\cos^{-1}z}\times$$
$$\times\Big[(2a+1)e^{\iota(2a\cos^{-1}z+\cos^{-1}z)}{}_2F_1\Big(1,1-\frac{1}{2a};2-\frac{1}{2a};-e^{2\iota a\cos^{-1}z}\Big)+$$
$$+(2a-1)\Big\{(2a+1)e^{\iota\cos^{-1}z}\Big({}_2F_1\Big(1,-\frac{1}{2a};1-\frac{1}{2a};-e^{2\iota a\cos^{-1}z}\Big)+$$
$$+e^{2\iota\cos^{-1}z}{}_2F_1\Big(1,\frac{1}{2a};1+\frac{1}{2a};-e^{2\iota a\cos^{-1}z}\Big)\Big)-$$
$$-e^{\iota(2a+3)\cos^{-1}z}{}_2F_1\Big(1,1+\frac{1}{2a};2+\frac{1}{2a};-e^{2\iota a\cos^{-1}z}\Big)\Big\}\Big]+Constant \tag{2.7}$$

$$\int \cot(a\cos^{-1}z)\,dz = -\frac{1}{2(4a^2-1)}\iota\;e^{-2\iota\cos^{-1}z}\times$$
$$\times\Big[(2a-1)\Big\{e^{\iota(2a+3)\cos^{-1}z}{}_2F_1\Big(1,1+\frac{1}{2a};2+\frac{1}{2a};e^{2\iota a\cos^{-1}z}\Big)+$$
$$+(2a+1)e^{\iota\cos^{-1}z}\Big({}_2F_1\Big(1,-\frac{1}{2a};1-\frac{1}{2a};e^{2\iota a\cos^{-1}z}\Big)+$$
$$+e^{2\iota\cos^{-1}z}{}_2F_1\Big(1,\frac{1}{2a};1+\frac{1}{2a};e^{2\iota a\cos^{-1}z}\Big)\Big)\Big\}-$$
$$-(2a+1)e^{\iota(2a\cos^{-1}z+\cos^{-1}z)}{}_2F_1\Big(1,1-\frac{1}{2a};2-\frac{1}{2a};e^{2\iota a\cos^{-1}z}\Big)\Big]+Constant \tag{2.8}$$

REFERENCES

1. Abramowitz, Milton; Stegun, Irene A. (1965), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, New York: Dover.
2. Ahlfors, Lars (1979), Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable (3rd ed.). New York; London: McGraw-Hill.
3. Alfred George Greenhill (1892), The applications of elliptic functions, New York, Macmil-lan.
4. Andrews, L.C. (1992); Special Function of mathematics for Engineers, Second Edition, McGraw-Hill Co Inc., New York.
5. Bells, Richard, Wong, Roderick; Special Functions, A Graduate Text, Cambridge Studies in Advanced Mathematics, 2010.
6. Borowski, Ephraim; Borwein, Jonathan (2002), Mathematics, Collins Dictionary (2^{nd} ed.).Glasgow: HarperCollins.
7. Crandall, Richard; Pomeramce, Carl (2001), Prime Numbers: A Computational Perspective, New York: Springer.
8. Derbyshire, J. (2004) Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics, New York: Penguin.
9. Graham, Ronald L.; Knuth, Donald E.; Patashnik, Oren (1994), Concrete Mathematics, Reading Ma: Addison-Wesley.
10. Harris Hancock (1910), Lectures on the theory of Elliptic functions, New York, J. Wiley &sons.
11. Krantz, S. G. (1999), Handbook of Complex Variables, Boston, MA: Birkhuser.

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