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REGIONS CONTAINING NO ZERO OF A POLYNOMIAL

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ABSTRACT

In this paper we find zero-free regions for complex polynomials under certain conditions on their coefficients.

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1. INTRODUCTION AND STATEMENT OF RESULTS

In the literature a lot of papers are available giving the regions which contain all or some of the zeros of a polynomial under certain coefficient conditions. Recently Sahu and Neha [2] proved the following result:

Theorem: A Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and for some real numbers $k_1,k_2,\tau_1,\tau_2;k_1\geq 1,k_2\geq 1;$

$$0<\tau_1\leq 1, 0<\tau_2\leq 1,$$
$$k_1\alpha_n\geq\alpha_{n-1}\geq......\geq\alpha_1\geq\tau_1\alpha_0$$
$$k_2\beta_n\geq\beta_{n-1}\geq......\geq\beta_1\geq\tau_2\beta_0.$$

Then all the zeros of P(z) lie in the disc

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\leq\frac{1}{|a_n|}[\{(k_1\alpha_n-\alpha_{n-1})^2+(k_2\beta_n-\beta_{n-1})^2\}^{\frac{1}{2}}+2(\tau_1|\alpha_0|+\tau_2|\beta_0|)$$
$$-(\tau_1|\alpha_0|-\tau_2|\beta_0|)]$$

In this paper we prove the following:

Theorem: 1 Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and for some real numbers $k_1,k_2,\tau_1,\tau_2;k_1\geq 1,k_2\geq 1;$

$$0<\tau_1\leq 1, 0<\tau_2\leq 1,$$
$$k_1\alpha_n\geq\alpha_{n-1}\geq......\geq\alpha_1\geq\tau_1\alpha_0$$
$$k_2\beta_n\geq\beta_{n-1}\geq......\geq\beta_1\geq\tau_2\beta_0.$$

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Then P(z) has no zero in $|z| < \frac{|a_0|}{M_1}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_2}$ for $R \le 1$,

where

$$M_1 = R^{n+1}[|a_n| + \{(k_1-1)^2\alpha_n{}^2 + (k_2-1)^2\beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|) - \tau_2(\beta_0 + |\beta_0|) + |\alpha_0| + |\beta_0|]$$

and

$$M_2 = R[|a_n| + \{(k_1-1)^2\alpha_n{}^2 + (k_2-1)^2\beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|) - \tau_2(\beta_0 + |\beta_0|) + |\alpha_0| + |\beta_0|]$$

Theorem: 2 Let Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and for some real numbers $k_1, k_2, \tau_1, \tau_2; k_1 \ge 1, k_2 \ge 1;$

$0 < \tau_1 \le 1, 0 < \tau_2 \le 1$,

$k_1\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \tau_1\alpha_0$

$k_2\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \tau_2\beta_0$.

Then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$,

where

$$M_3 = R^{n+1}[|a_n| + \{(k_1-1)^2\alpha_n{}^2 + (k_2-1)^2\beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|) - \tau_2(\beta_0 + |\beta_0|) + 2(|\alpha_0| + |\beta_0|)] \text{ for } R \ge 1$$

and the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$,

where

$$M_4 = |a_0| + R[|a_n| + \{(k_1-1)^2\alpha_n{}^2 + (k_2-1)^2\beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|) - \tau_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \le 1.$$

Remark: Combining Theorem 1 and Theorem 2, we get the following result:

Theorem: 3 Let Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and for some real numbers $k_1, k_2, \tau_1, \tau_2; k_1 \ge 1, k_2 \ge 1;$

$0 < \tau_1 \le 1, 0 < \tau_2 \le 1$,

$k_1\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \tau_1\alpha_0$

$k_2\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \tau_2\beta_0$.

Then the number of zeros of P(z) in $\frac{|a_0|}{M_3} \le |z| \le \frac{R}{c} (R > 0, c > 1), j = 1,2$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$, and the number of zeros of P(z) in $\frac{|a_0|}{M_4} \le |z| \le \frac{R}{c} (R > 0, c > 1), j = 1,2$ does not exceed $\frac{1}{\log c}\log\frac{M_2}{|a_0|}$, where M_1, M_2, M_3, M_4 are as given in Theorems 1 and 2.

For different values of the parameters we get various interesting results from the above theorems.

2. LEMMAS

For the proofs of the above results we need the following results:

Lemma: 1 If f(z) is analytic in $|z| \leq R,$ but not identically zero, f(0)$\neq$ 0 and $f(a_k)=0, k=1,2,\ldots\ldots,n$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(\mathrm{Re}^{i\theta}\right|d\theta-\log|f(0)|=\sum_{j=1}^{n}\log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's theorem (see page 208 of [1]).

Lemma: 2 If f(z) is analytic and $|f(z)| \leq M(r)$ in $|z| \leq r,$ then the number of zeros of f(z) in $|z| \leq \frac{r}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{M(r)}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 1.

3. PROOFS OF THEOREMS

Proof of Theorem: 1 Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots\ldots\ldots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + (k_1\alpha_n - \alpha_{n-1})z^n - (k_1 - 1)\alpha_n z^n + \ldots\ldots \\
&\quad + i(k_2\beta_n - \beta_{n-1})z^n - i(k_2 - 1)\beta_n z^n + \ldots\ldots + (\alpha_1 - \tau_1\alpha_0)z \\
&\quad + (\tau_1 - 1)\alpha_0 z + i(\beta_1 - \tau_2\beta_0)z + i(\tau_2 - 1)\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + a_0 \\
&= G(z) + a_0, \text{ where}
\end{aligned}$$

$$\begin{aligned}
G(z) &= -a_n z^{n+1} + (k_1\alpha_n - \alpha_{n-1})z^n - (k_1 - 1)\alpha_n z^n + \ldots\ldots \\
&\quad + i(k_2\beta_n - \beta_{n-1})z^n - i(k_2 - 1)\beta_n z^n + \ldots\ldots + (\alpha_1 - \tau_1\alpha_0)z \\
&\quad + (\tau_1 - 1)\alpha_0 z + i(\beta_1 - \tau_2\beta_0)z + i(\tau_2 - 1)\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j.
\end{aligned}$$

For $|z| = R,$ we have, by using the hypothesis,

$$\begin{aligned}
|G(z)| &\leq |a_n|R^{n+1} + |(k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n|R^n + (k_1\alpha_n - \alpha_{n-1})R^n + (k_2\beta_n - \beta_{n-1})R^n \\
&\quad + (\alpha_{n-1} - \alpha_{n-2})R^{n-1} + \ldots\ldots + (\alpha_1 - \tau_1\alpha_0)R + (1-\tau_1)|\alpha_0|R \\
&\quad + (k_2\beta_n - \beta_{n-1})R^n + (\beta_{n-1} - \beta_{n-2})R^{n-1} + \ldots\ldots + (\beta_1 - \tau_2\beta_0)R + (1-\tau_2)|\beta_0|R
\end{aligned}$$

For $R \geq 1$,

$$\begin{aligned}
|G(z)| &\leq R^{n+1}[|a_n| + \{(k_1 - 1)^2\alpha_n{}^2 + (k_2 - 1)^2\beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|) \\
&\quad - \tau_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)], \\
&= M_1,
\end{aligned}$$

and for $R \le 1$,

$$\begin{aligned}|G(z)| &\le R[|a_n| + \{(k_1-1)^2 \alpha_n{}^2 + (k_2-1)^2 \beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|) \\ &\quad -\tau_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)]. \\ &= M_2.\end{aligned}$$

Since G(z) is analytic for $|z| \le R,$ G(0)=0, it follows by Schwarz Lemma that

$|G(z)| \le M_1|z|,$ for $|z| \le R,\ R \ge 1$ and $|G(z)| \le M_2|z|,$ for $|z| \le R,\ R \le 1$.

Therefore, for $|z| \le R,\ R \ge 1$,

$$\begin{aligned}|F(z)| &= |a_0 + G(z)| \\ &\ge |a_0| - |G(z)| \\ &\ge |a_0| - M_1|z| \\ &> 0 \text{ if}\end{aligned}$$

$$|z| < \frac{|a_0|}{M_1}.$$

And, for $|z| \le R,\ R \le 1$,

$$\begin{aligned}|F(z)| &= |a_0 + G(z)| \\ &\ge |a_0| - |G(z)| \\ &\ge |a_0| - M_1|z| \\ &> 0 \text{ if}\end{aligned}$$

$$|z| < \frac{|a_0|}{M_2}.$$

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{M_1}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_2}$ for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), it follows that P(z) has no zero in $|z| < \frac{|a_0|}{M_1}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_2}$ for $R \le 1$ and the theorem is proved.

Proof of Theorem: 2 Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots\ldots\ldots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (k_1\alpha_n - \alpha_{n-1})z^n - (k_1-1)\alpha_n z^n + \ldots\ldots \\ &\quad + i(k_2\beta_n - \beta_{n-1})z^n - i(k_2-1)\beta_n z^n + \ldots\ldots + (\alpha_1 - \tau_1\alpha_0)z \\ &\quad + (\tau_1 - 1)\alpha_0 z + i(\beta_1 - \tau_2\beta_0)z + i(\tau_2 - 1)\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + a_0\end{aligned}$$

For $|z| \le R,$ we have, by using the hypothesis,

$$|F(z)| \le |a_n|R^{n+1} + |(k_1-1)\alpha_n + i(k_2-1)\beta_n|R^n + (k_1\alpha_n - \alpha_{n-1})R^n + (k_2\beta_n - \beta_{n-1})R^n$$
$$+ (\alpha_{n-1} - \alpha_{n-2})R^{n-1} + ... + (\alpha_1 - \tau_1\alpha_0)R + (1-\tau_1)|\alpha_0|R + |a_0|$$
$$+ (k_2\beta_n - \beta_{n-1})R^n + (\beta_{n-1} - \beta_{n-2})R^{n-1} + ... + (\beta_1 - \tau_2\beta_0)R + (1-\tau_2)|\beta_0|R$$

For $R \ge 1$,

$$|F(z)| \le R^{n+1}[|a_n| + \{(k_1-1)^2\alpha_n{}^2 + (k_2-1)^2\beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|)$$
$$- \tau_2(\beta_0 + |\beta_0|) + 2(|\alpha_0| + |\beta_0|)],$$
$$= M_3,$$

and for $R \le 1$,

$$|F(z)| \le |a_0| + R[|a_n| + \{(k_1-1)^2\alpha_n{}^2 + (k_2-1)^2\beta_n{}^2\}^{\frac{1}{2}} + (k_1\alpha_n + k_2\beta_n) - \tau_1(\alpha_0 + |\alpha_0|)$$
$$- \tau_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)].$$
$$= M_4 \quad .$$

Since F(z) is analytic for $|z| \le R,$ it follows by Lemma 2 that the number of zeros of F(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_n|}$ for $R \ge 1$ and the number of zeros of F(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_n|}$ for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), the result follows.

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