International Research Journal of Pure Algebra -3(11), 2013, 340-342
Available online through www.rjpa.info ISSN 2248–9037

POLYNOMIAL MATRIX FULL RANK DECOMPOSITION AND ITS APPLICATIONS

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(Received on: 26-11-13; Revised & Accepted on: 30-11-13)

ABSTRACT

In this paper, we have proved the equivalence of minor left prime and factor left prime and the existence of full rank decomposition for matrices over $K^{s\times t}[X]$.Moreover we have given a application of full rank decomposition, namely, some non-singular matrix is shift equivalent to every non-nilpotent matrix over $K^{s\times t}[X]$.

Keywords: *polynomial matrix; full rank decomposition; prime matrix.*

1. INTRODUCTION

Let A is a polynomial matrix of field, $A \in F^{s\times t}[x_1, x_2 \cdots x_n]$, and $rankA = r$. Now, is there a full rank decomposition to make $A = A_1A_2, A_1 \in F^{s\times r}[x_1, x_2 \cdots x_n], A_2 \in F^{r\times t}[x_1, x_2 \cdots x_n]$. $D.C.Youla$ have indicated that the result hold up when $n \le 2$, but the result didn't establish when $n \ge 3$ in reference [3]. By studying the equivalent between minor left prime and factor left prime, we can certify the existence of full rank decomposition for matrices over $K^{s\times t}[X]$.

2. PREPARATION KNOWLEDGE

Definition: 1[1] Let K is a skew field, $K[x_1, x_2 \cdots x_n]$ represents a polynomial ring which has n variable, denoted by $K[X]$. $K^{s\times t}[x_1, x_2 \cdots x_n]$ represents the whole $s\times t$ order matrix in $K[X]$, denoted by $K^{s\times t}[X]$. Every matrix in $K^{s\times t}[X]$ is called polynomialmatrix.

Definition: 2[1] As to m order square matrix M, $\det M = \dfrac{M^*}{M^{-1}}$ is a polynomial. If $\det M \ne 0$, then M is called unimodular matrix.

Definition: 3[2] Let $A \in K^{s\times t}[X]$, if the greatest common divisor of $s\times s$ order minor of A is a invertible element, we can call A is a minor left prime matrix
As to the whole polynomial matrix decomposition of A, $A = A_1A_2$. If A_1 is a square matrix, then A_2 is unimodular matrix. We call A is a factor left prime.

Definition: 4[5] Let R is a associative rings, square matrix $A, B \in R$. If there is a positive integer l and $B_i, C_i \in R$,then $A = B_1C_1, C_1B_1 = B_2C_2, C_2B_2 = B_3C_3 \cdots B_lC_l = B$,we say A is shift equivalent to B.

Lemma: 1[4] Let $A \in K^{s\times t}[X]\,(s \le t)$, $d(x)$ is the greatest common divisor of the highest order minor of A, there is a decomposition $A = HL$ such that $\det H = d(x)$, $H \in K^{s\times s}[x]$, $L \in K^{s\times t}[x]$.

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Lemma: 2[4] Let $A \in k^{s\times t}\left[X\right]\left(s \le t\right)$,we can rewrite A as $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$,where A is a non-singular and $A_{22}A_{11}^{-1}A_{12}$ is a polynomial matrix. If (A_{11}, A_{12}) is a factor left prime, then $A_{22}A_{11}^{-1}$ is a polynomial matrix.

Lemma: 3 As to the matrices of $K^{s\times t}\left[X\right]$, minor left prime matrix is equivalent to factor left prime matrix.

Proof: Minor left prime matrix $\Rightarrow$ factor left prime matrix

Let $A \in K^{s\times t}\left[X\right]\left(s \le t\right)$is a minor left prime matrix, $d(x)$ is the greatest common divisor of the highest order minor of A. If there is a decomposition $A = A_1 A_2$, $A_1 \in K^{s\times s}[X]$, $A_2 \in K^{s\times t}[X]$. Then $d(x)$ can be divisible by A_1. According to A is a minor left prime matrix, we can learn $d(x)$ is a invertible element. So, $\det A_1$ is a invertible element. Thus, A_1 is a unimodular matrix. So A is a factor left prime.factor left prime matrix $\Rightarrow$ Minor left prime matrix

Let $A \in K^{s\times t}\left[X\right]\left(s \le t\right)$is a factor left prime matrix. $d(x)$ is the greatest common divisor of the highest order minor of A. According to lemma 1,there is a decomposition $A = A_1 A_2$ such that $\det A_1 = d(x)$. Because of A is a factor left prime matrix, then $d(x)$ is a invertible element. So A is a minor left prime matrix.

3. MAIN CONCLUSION

Theorem: 1 Let $A \in K^{s\times t}[X]$ $\left(s \le t\right)$, $rankA = r < s$. Then, there is a decomposition such that $A = A_1 A_2$, $A_1 \in K^{s\times r}[x], A_2 \in K^{r\times t}[x]$.

Proof: As to arbitrarily matrix A, by the transformation of row and line, A can be written as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$\det A \ne 0,\ A_{11} \in K^{r\times r}[X], A_{12} \in K^{r\times(t-r)}[X], A_{21} \in K^{(s-r)\times r}[X], A_{22} \in K^{(s-r)\times(t-r)}[X], A_{22} = A_{21}A_{11}^{-1}A_{12}$.

Let $d(x)$ is the greatest common divisor of $r\times r$ order minor matrix of A. According to lemma 1, there is a decomposition $(A_{11}, A_{12}) = H(L_{11}, L_{12})$, $H \in K^{r\times r}[X]$, $L_{11} \in K^{r\times r}[X]$, $L_{12} \in K^{r\times(t-r)}[X]$, then $\det H = d(x)$, (L_{11}, L_{12}) is a factor left prime matrix. According to lemma 3, (L_{11}, L_{12}) is a minor left prime. Because of $A_{21}L_{11}^{-1}L_{12}$ $= A_{21}(HL_{11})^{-1}(HL_{12}) = A_{21}A_{11}^{-1}A_{22}$ is a polynomial matrix, then $A_{21}A_{11}^{-1}$ is a polynomial matrix. So there is a decomposition as follows:

$$\begin{pmatrix} L_{11} & L_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I_r \\ A_{21}L_{11}^{-1} \end{pmatrix}\begin{pmatrix} L_{11} & L_{12} \end{pmatrix}$$

$$A = \begin{pmatrix} H & 0 \\ 0 & I_{m-r} \end{pmatrix}\begin{pmatrix} L_{11} & L_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I_{m-r} \end{pmatrix}\begin{pmatrix} I_r \\ A_{21}A_{11}^{-1} \end{pmatrix}\begin{pmatrix} L_{11} & L_{12} \end{pmatrix} = \begin{pmatrix} H \\ A_{21}L_{11}^{-1} \end{pmatrix}\begin{pmatrix} L_{11} & L_{12} \end{pmatrix}$$
$$= A_1 A_2$$
$$A_1 = \begin{pmatrix} H \\ A_{21}L_{11}^{-1} \end{pmatrix} \in K^{s\times r}[X], A_2 = \begin{pmatrix} L_{11} & L_{12} \end{pmatrix} \in K^{r\times t}[X].$$

Theorem: 2 Let $A \in K^{n\times n}[X]$, $rankA = r < n$, then, there is a non-singular square matrix B such that A is shift equivalent to B.

Proof: According to theorem 1, there is a decomposition $A = B_1C_1$, $B \in K^{n\times r}[X]$, $C \in K^{r\times n}$, $rankB_1 = rankC_1 = r$, $C_1B_1 \in K^{r\times r}[X]$. If C_1B_1 is a non-singular square matrix, then suppose $B = C_1B_1$, and $A^2 = B_1(C_1B_1)C_1$, $rankA^2 = rank(C_1B_1) = r = rankA$.

If $rank(C_1B_1) = r_1 < r$, then we can continue full rank decomposition to C_1B_1, we can obtain decomposition sequence $A = B_1C_1, C_1B_1 = B_2C_2, C_2B_2 = B_3C_3 \cdots$, B_iC_i and $C_iB_i\,(i = 1,2,\cdots)$ are square matrix. Noting the order of C_iB_i is strict smaller than $C_{i-1}B_{i-1}$, and n is limited, so there must be a positive integer number such that C_iB_i is a non-singular square matrix. Because:

$$A^k = (B_1C_1)^k = B_1(C_1B_1)^{k-1}C_1 = \cdots = B_1C_2\cdots B_{k-1}(B_kC_k)C_{k-1}\cdots C_2C_1$$

$$A^{k+1} = B_1C_2\cdots B_{k-1}B_k(C_kB_k)C_kC_{k-1}\cdots C_2C_1$$

If A a non-singular square matrix, then $A^{k+1} \neq 0$, so $C_kB_k \neq 0$. Suppose $B_kC_k = B$, then B a non-singular square matrix, and A is shift equivalent to B.

Let $B_k \in K^{p\times r}[X], C_k \in K^{r\times p}[X]$, according to full rank decomposition above, we know that $rankB_kC_k = r$, because of $C_kB_k \in K^{r\times r}[X]$ and C_kB_k a non-singular square matrix, so $rankB_kC_k = r = rank(C_kB_k)$. According to the analysis above, we can obtain the conclusion as follows:

$$rankA^{k+1} = rank(C_kB_k) = rank(B_kC_k) = rankA^k$$

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Source of Support: Nil, Conflict of interest: None Declared