International Research Journal of Pure Algebra -3(11), 2013, 343-346
Available online through www.rjpa.info ISSN 2248–9037

SUPER FILTERS OF B-ALMOST DISTRIBUTIVE LATTICES

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(Received on: 12-11-13; Revised & Accepted on: 28-11-13)

ABSTRACT

Different properties of Super filters of $B-$Almost Distributive Lattices are derived. Basic facts of super filters of $B-$Almost Distributive Lattices are obtained. Different necessary and sufficient conditions of super filters of $B-$Almost Distributive Lattices are derived.

AMS 2000 Subject Classification: *06D99.*

Keywords: *Almost Distributive Lattice (ADL); Birkhoff Center; Filter; Super filter; Maximal element; $B-$Almost Distributive Lattice($B-$ADL).*

1. INTRODUCTION

The concept of an Almost Distributive Lattice (ADL) was introduced by U.M. Swamy and G.C. Rao [6] as a common abstraction of most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. The concept of a Birkhoff center B of an ADL A was introduced in [7] and it was observed that B is a relatively complemented ADL. In [4], G. Epstein and A. Horn introduced theconcept of a $B-$algebra as a bounded distributive lattice with center B in which, for any $x, y \in A,$ the largest element $x \Rightarrow y$ in B exists with the property $x \wedge (x \Rightarrow y) \leq y.$ The connective $x \Rightarrow y$ of a $B-$algebra has several applications in logic and computer science [2,3]. For this reason, in [5], we introduced the concept a $B-$Almost Distributive Lattice ($B-$ADL) as an ADL in which the lattice of all principal ideals of A is a $B-$Algebra. In this paper, we introduce the concept of super filters of a $B-$ADLs and derive some basic properties of super filters of $B-$ADLs. Also, we obtain different characterizations of super filers of $B-$ADLs.

2. PRELIMINARIES

In this section, we give the necessary definitions and important properties of an ADL taken from [6] for ready reference.

Definition: 2.1[6] An algebra $(A, \vee, \wedge, 0)$ of type (2, 2, 0) is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms:

i. $x \vee 0 = x$
ii. $0 \wedge x = 0$
iii. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
iv. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
v. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
vi. $(x \vee y) \wedge y = y$ for all $x, y, z \in A.$

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Theorem: 2.2 [6] Let m be a maximal element in an ADL A and $x \in A.$ Then the following are equivalent:

i. x is a maximal element of $(A, \leq).$

ii. $x \wedge m = m.$

iii. $x \wedge a = a,$ for all $a \in A.$

iv. $x \vee a = x,$ for all $a \in A.$

v. $a \vee x$ is maximal, for all $a \in A.$

Definition: 2.3 [6]A non-empty subset I of an ADL A is called an ideal of A. If $x \vee y \in I$ and $x \wedge a \in I$ for any $x, y \in I$ and $a \in A.$ The principal ideal of A generated by x is denoted by $(x].$ The set $\mathrm{PI}(\mathrm{A})$ of all principal ideals of A forms a distributive lattice under the operations $\vee, \wedge$ defined by $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$ in which $(0]$ is the least element. If A has a maximal element $m,$ then $(m]$ is the greatest element of $PI(A).$

Definition: 2.4 [6] A non-empty subset F of an ADL A is called a filter if and only if it satisfies the following:

i. $x, y \in F \Rightarrow x \wedge y \in F.$

ii. $x \in F, a \in A \Rightarrow a \vee x \in F.$

For other properties of an ADL, we refer to [6].

The concept of Birkhoff Center of an Almost Distributive Lattice is introduced by U.M. Swamy and S. Ramesh in [7]. The following definition is taken from [7].

Definition: 2.5 [7] Let A be an ADL with a maximal element m and $B(A) = \{x \in A \mid x \wedge y = 0$ and $x \vee y$ is maximal for some $y \in A\}.$ Then $(B(A), \vee, \wedge)$ is a relatively complemented ADL and it is called the Birkhoff center of $A.$ We use the symbol B instead of $B(A)$ when there is no ambiguity.

For other properties of Birkhoff center of an ADL, we refer to [7].

In our paper [5], we introduced the concept of a $B-$Almost Distributive Lattice (or, simply a $B-$ADL) and studied its properties. The following definition is taken from [5].

Definition: 2.6[5] An ADL $(A, \vee, \wedge, 0)$ with a maximal element m and Birkhoff center B is called a $B-$ADL if for any $x, y \in A,$ there exists $b \in B$ such that

i. $y \wedge x \wedge b = x \wedge b.$

ii. If $c \in B$ such that $y \wedge x \wedge c = x \wedge c,$ then $b \wedge c = c$ and in this case, $b \wedge m$ is denoted by $x \Rightarrow y.$

The following theorems, $B-$ADLs are taken from [5] which are required to characterize super filters of B-ADLs.

Theorem: 2.7 [5] Let A be a $B-$ADL with a maximal element m and Birkhoff center $B.$ Then, for any $x, y \in A,$ we have the following:

i. $y \wedge x \wedge (x \Rightarrow y) = x \wedge (x \Rightarrow y)$ and consequently, $x \wedge (x \Rightarrow y) \leq y \wedge m.$

ii. If $c \in B, x \wedge c \wedge m \leq y \wedge m,$ then $c \wedge m \leq x \Rightarrow y.$

iii. $x \wedge m \leq y \wedge m$ if and only if $x \Rightarrow y = m.$

Theorem: 2.8 [5] Let A be a $B-$ADL with a maximal element m and Birkhoff center $B.$ Then, for any $x, y, z \in A$ and $a \in B,$ we have the following:

i. $x \wedge (x \Rightarrow a) = x \wedge a \wedge m.$

ii. $x \wedge (x \Rightarrow (y \Rightarrow z)) = x \wedge (y \Rightarrow z).$

iii. $a \wedge (x \Rightarrow a) = a \wedge m.$

iv. $(x \Rightarrow a) \wedge a = a.$

v. If $x \wedge m \leq y \wedge m,$ then $(z \Rightarrow x) \leq (z \Rightarrow y)$ and $(x \Rightarrow z) \geq (y \Rightarrow z).$

For other properties of a $B-$ ADLs, we refer to [5].

3. SUPER FILTERS OF B-ADLs

Definition 3.1 Let A be a $B-$ ADL with a maximal element m and Birkhoff center $B.$ Suppose S is a non-empty subset of $A.$ Then S is said to be a super filter of a $B-$ ADL $A,$ if it satisfies the following conditions: for all $x, y \in A;$

$S_1:$ If $x, y \in S,$ then $x \wedge y \in S.$

$S_2:$ If $x \in S$ and $x \wedge m \leq y \wedge m,$ then $y \in S.$

Example: 3.2 Let A be a discrete ADL with 0 and with at least two elements and Birkhoff center $B.$ Fix $m(\neq 0) \in A$ and define for any $x, y \in A,$

$$(x \Rightarrow y) = 0 \text{ if } x \neq 0,\ y=0$$
$$= m \text{ otherwise.}$$

Then $(A, \vee, \wedge, \Rightarrow, 0, m)$ is a $B-$ ADL and $\{m\}$ is a super filter in $A.$

Theorem: 3.3 Let A be a $B-$ ADL with a maximal element m and Birkhoff center $B.$ Then every filter of a A containing m is a super filter of $A.$

Proof: Let S be a filter of $B-$ ADL A containing $m.$ Then, for any $x, y \in S,$ we get $x \wedge y \in S.$ Let $x \in S$ and $x \wedge m \leq y \wedge m.$ Then $x \wedge m \in S$ (since $m \in S$). Now $y \wedge x \wedge m = x \wedge m \wedge y \wedge m = x \wedge m.$
Then $y \wedge x \wedge m \in S.$ Now $y = y \vee (y \wedge x \wedge m) \in S.$ Therefore S is a super filter of $A.$

Theorem: 3.4 Let A be a $B-$ ADL with a maximal element m and Birkhoff center $B.$ Suppose S is a non-empty subset of $A.$ Then S is a super filter of A if and only if it satisfies the following conditions:

i. $m \in S.$

ii. If $x \in S, (x \Rightarrow y) \in S,$ then $y \in S$ for all $x, y \in B.$

Proof: Suppose S is a super filter of A and $x, y \in B.$ Then, clearly $m \in S.$ Let $x \in S$ and $(x \Rightarrow y) \in S.$ Then $x \wedge (x \Rightarrow y) \in S.$ But $x \wedge (x \Rightarrow y) = x \wedge y \wedge m = y \wedge x \wedge m \in S.$ Since S is a filter and $y \wedge x \wedge m \in S,$ $y \in A,$ we get that $y \vee (y \wedge x \wedge m) = y \in S.$ Conversely, suppose conditions (i) and (ii) hold. Let $x, y \in S.$ Since $y \in B,$ we have

$y \wedge m \leq (x \Rightarrow y)$ implies $y \wedge m \leq (x \Rightarrow (x \wedge y))$ (since $(x \Rightarrow y) = (x \Rightarrow (x \wedge y))$

implies $(y \Rightarrow y) \leq (y \Rightarrow (x \Rightarrow (x \wedge y)))$

implies $m = (y \Rightarrow (x \Rightarrow (x \wedge y))) \in S.$

Since $y \in S, (y \Rightarrow (x \Rightarrow (x \wedge y))) \in S$ and $y \in B, (x \Rightarrow (x \wedge y)) \in B$ by our assumption, we get $(x \Rightarrow (x \wedge y)) \in S.$ Again, since $x \in S, (x \Rightarrow (x \wedge y)) \in S$ and $x \in B,$ $x \wedge y \in B$ by our assumption, we get $x \wedge y \in S.$ Let $x \in S$ and $x \wedge m \leq y \wedge m.$ Then $(x \Rightarrow y) = m \in S.$ Since $x \in S, (x \Rightarrow y) \in S$ and $x \in B,$ $y \in B$ we get $y \in S.$ Therefore S is a super filter of $A.$

Theorem: 3.5 Let A be a $B-$ ADL with a maximal element m and Birkhoff center $B.$ Suppose S is a non-empty subset of A and $x, y, z \in B.$ Then S is a super filter of A if and only if $x \wedge m \leq (y \Rightarrow z)$ implies $z \in S$ for all $x, y \in S$ and $z \in A.$

Proof: Suppose S is a super filter of $A.$ Let $x \wedge m \leq (y \Rightarrow z)$ for all $x, y \in S$ and $z \in A.$ Then $(x \Rightarrow x) \leq (x \Rightarrow (y \Rightarrow z))$ and hence $m = (x \Rightarrow (y \Rightarrow z)) \in S.$ Since $x \in S, (x \Rightarrow (y \Rightarrow z)) \in S,$ by Theorem 3.4, we get $(y \Rightarrow z) \in S.$ Again, since $y \in S, (y \Rightarrow z) \in S,$ by Theorem 3.4, we get that $z \in S.$

Conversely, suppose $x \wedge m \leq (y \Rightarrow z)$ implies $z \in S$ for all $x, y \in S$ and $z \in A$. Let $x, y \in S$. Since $y \in B$, we have $y \wedge m \leq (x \Rightarrow y)$ and hence $y \wedge m \leq (x \Rightarrow (x \wedge y))$. By our assumption, we get $x \wedge y \in S$. Let $x \in S$ and $x \wedge m \leq y \wedge m$. Then $(x \Rightarrow x) \leq (x \Rightarrow y)$ and hence $m \leq (x \Rightarrow y)$. Thus, by our assumption, we get $y \in S$. Hence S is a super filter of A.

The following corollary is direct consequence of the above theorem.

Corollary: 3.6 Let A be a $B-$ADL with a maximal element m and Birkhoff center B. Suppose S is a non-empty subset of A and $x, y, z \in B$. Then S is a super filter of A if and only if $(x \Rightarrow (y \Rightarrow z)) = m$ implies $z \in S$ for all $x, y \in S, z \in A$.

Theorem: 3.7 Let A be a $B-$ADL with a maximal element m and Birkhoff center B. Suppose S is a non-empty subset of A and $y \in S$. Then $((x \Rightarrow y) \Rightarrow z) \in S$ implies $(x \Rightarrow (y \Rightarrow z)) \in S$ *for all* $x \in A$ *and* $y, z \in B$.

Proof: Let S be a super filter of A. Suppose $((x \Rightarrow y) \Rightarrow z) \in S$. Since $y \in B$, we have $y \wedge m \leq (x \Rightarrow y)$ and hence $(y \Rightarrow z) \geq ((x \Rightarrow y) \Rightarrow z)$. Thus $((x \Rightarrow y) \Rightarrow z) \Rightarrow (y \Rightarrow z) = m$. By Corollary 3.6, we get $(y \Rightarrow z) \in S$. Since $(y \Rightarrow z) \leq (x \Rightarrow (y \Rightarrow z))$, we have $(y \Rightarrow z) \Rightarrow (x \Rightarrow (y \Rightarrow z)) = m \in S$. Since $(y \Rightarrow z) \in S$, $(y \Rightarrow z) \Rightarrow (x \Rightarrow (y \Rightarrow z)) \in S$, by Theorem 3.4, we get $(x \Rightarrow (y \Rightarrow z)) \in S$.

Finally, we conclude this paper with the following.

Theorem: 3.8 Let A be a $B-$ADL with a maximal element m and Birkhoff center B. Suppose S is a non-empty subset of A and $y, z \in S$. Then $((x \Rightarrow z) \Rightarrow y) \in S$ implies $x \Rightarrow y \in S$ for all $x \in A$ and $y, z \in B$.

Proof: Suppose $((x \Rightarrow z) \Rightarrow y) \in S$ for all $x \in A$ and $y, z \in B$. Since $z \in B$, we have $z \wedge m \leq (x \Rightarrow z)$ and hence $(z \Rightarrow (x \Rightarrow z)) = m \in S$. Since $z \in S$, by Theorem 3.4, we get $(x \Rightarrow z) \in S$. Again, since $(x \Rightarrow z) \in S$, by Theorem 3.4, we get $y \in S$. Since $y \in B$ and $y \wedge m \leq (x \Rightarrow y)$ and hence $(y \Rightarrow (x \Rightarrow y)) = m \in S$. Since $y \in S$, by Theorem 3.4, we get $(x \Rightarrow y) \in S$.

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Source of Support: Nil, Conflict of interest: None Declared