



SOME RESULTS ON THE GROUP INVERSE OF BLOCK MATRIX OVER SKEW FIELDS

Junqing Wang* & Xin Fan

Department of Mathematics, Dezhou University, Shandong 253055, PR China.

(Received on: 04-12-13; Revised & Accepted on: 18-12-13)

ABSTRACT

In this paper, we give the existences and the representations of the group inverse for block matrix $\begin{bmatrix} AC & B \\ C & 0 \end{bmatrix}$ and $\begin{bmatrix} AB & B \\ C & 0 \end{bmatrix}$ under the special condition.

Keywords: block matrix; skew fields; group inverse.

1. INTRODUCTION

Let K is a skew fields, $K^{m \times n}$ represents the set of $m \times n$ matrices over K . For $A \in K^{m \times n}$, if X satisfying $AXA = X$, $XAX = X$, $AX = XA$. We call X is the group inverse A , we write $X = A^\#$. In this paper, $A^\pi = I - AA^\#$.

Some authors and experts have given the representations of the group inverses in [1] and [2]. Actually, generalized inverses have wide applications in many areas such as special matrix theory, singular differential and difference equations and graph theory. The presentations of block matrix $\begin{bmatrix} AC & B \\ C & D \end{bmatrix}$ is gives in [3] under the assumption that

A and $I - CA^{-1}B$ are both invertible over any fields; however, the representation of the group inverse $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is not given. In this paper, we obtain the representation of the block matrix $\begin{bmatrix} AC & B \\ C & 0 \end{bmatrix}$ and $\begin{bmatrix} AB & B \\ C & 0 \end{bmatrix}$.

2. PREPARATION KNOWLEDGE

In order to prove the main results, we give some lemmas.

Lemma: 1^[1] Let $A \in K^{m \times n}$, then $A^\#$ exists if and only if $\text{rank}A = \text{rank}A^2$.

Lemma: 2^[1] Let $A, B \in K^{m \times n}$, if $\text{rank}A = \text{rank}B = \text{rank}(AB) = \text{rank}(BA)$, 则, $(AB)^\#$ and $(BA)^\#$ exists, then the following conclusions hold:

- (1) $AB(AB)^\#A = A$
- (2) $A(BA)^\#BA = A$
- (3) $BA(BA)^\#B = B$
- (4) $B(AB)^\#A = BA(BA)^\#$
- (5) $A(BA)^\# = (AB)^\#A$

*Corresponding author: Junqing Wang

Department of Mathematics, Dezhou University, Shandong 253055, PR China.

Lemma: ^{3[2]} Let $A, X \in K^{n \times n}$, then $X = A^\#$ if and only if $AXA = A, AX = XA, \text{rank}X \leq \text{rank}A$.

3. MAIN CONCLUSION

Theorem: 1 Let $M = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} \in K^{2l \times 2l}$, and $\text{rank}B \geq \text{rank}C$, then:

- (1) $M^\#$ exists if and only if $\text{rank}B = \text{rank}C = \text{rank}(BC) = \text{rank}(CB)$
- (2) If $M^\#$ exists, then

$$M^\# = \begin{bmatrix} (BC)^\tau AC(BC)^\# & (BC)^\# - (BC)^\tau [AC(BC)^\#]^\# B \\ C(BC)^\# & -C(BC)^\# AC(BC)^\# B \end{bmatrix}$$

Proof: Because

$$\text{rank}M = \text{rank} \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = \text{rank}B + \text{rank}C$$

$$\text{rank}M^2 = \text{rank} \begin{bmatrix} ACAC + BC & ACB \\ CAC & CB \end{bmatrix} = \text{rank} \begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix}$$

First, we prove the ‘if’ part

According to the lemma 2, if $\text{rank}B = \text{rank}C = \text{rank}(BC) = \text{rank}(CB)$, then $CB(CB)^\# C = C$,

So $CAC = CB(CB)^\# CAC$, we get

$$\text{rank} \begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix} = \text{rank} \begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix} = \text{rank}(BC) + \text{rank}(CB)$$

So, $\text{rank}M = \text{rank}B + \text{rank}C = \text{rank}BC + \text{rank}CB = \text{rank}M^2$.

Second, we prove ‘only if’ part

Science $M^\#$ exists if and only if $\text{rank}M = \text{rank}M^2$, we have

$$\begin{aligned} \text{rank}B + \text{rank}C &= \text{rank} \begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix} = \text{rank} \left(\begin{bmatrix} BC & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & 0 \\ AC & B \end{bmatrix} \right) \\ &\leq \begin{pmatrix} BC & 0 \\ 0 & C \end{pmatrix} = \text{rank}(BC) + \text{rank}C \end{aligned}$$

$$\begin{aligned} \text{rank}B + \text{rank}C &= \text{rank} \begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix} = \text{rank} \left(\begin{bmatrix} B & 0 \\ AC & I \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & CB \end{bmatrix} \right) \\ &\leq \begin{pmatrix} C & 0 \\ 0 & CB \end{pmatrix} = \text{rank}C + \text{rank}(BC) \end{aligned}$$

Then $\text{rank}B \leq \text{rank}(BC), \text{rank}B \leq \text{rank}(CB)$, then

$$\text{rank}B = \text{rank}(BC) = \text{rank}(CB)$$

Because $\text{rank}B \geq \text{rank}(C)$, then $\text{rank}B = \text{rank}C = \text{rank}(BC) = \text{rank}(CB)$.

(2) If $M^\#$ exists, $G = \begin{bmatrix} (BC)^\tau AC(BC)^\# & (BC)^\# - (BC)^\tau [AC(BC)^\#]^\sharp B \\ C(BC)^\# & -C(BC)^\# AC(BC)^\# B \end{bmatrix}$

Write $G = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$, we can obtain,

$$MG = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} ACX + BZ & ACY + BW \\ CX & CY \end{bmatrix}$$

$$GM = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} XAC + YC & XB \\ ZAC + WC & ZB \end{bmatrix}$$

$$(MG)_{11} = (GM)_{11} = BC(BC)^\# \quad (MG)_{12} = (GM)_{12} = (BC)^\tau AC(BC)^\# B$$

$$(MG)_{21} = (GM)_{21} = 0 \quad (MG)_{22} = (GM)_{22} = C(BC)^\# B$$

Then $MG = GM = \begin{bmatrix} BC(BC)^\# & (BC)^\tau AC(BC)^\# B \\ 0 & C(BC)^\# B \end{bmatrix}$

Because lemma 2, we can obtain:

$$\begin{aligned} MGM &= \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} \begin{bmatrix} BC(BC)^\# & (BC)^\tau AC(BC)^\# B \\ 0 & C(BC)^\# B \end{bmatrix} \\ &= \begin{bmatrix} ACBC(BC)^\# & AC(BC)^\tau AC(BC)^\# B + BC(BC)^\# B \\ CBC(BC)^\# & C(BC)^\tau AC(BC)^\# B \end{bmatrix} = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = M. \end{aligned}$$

According to the same theory, $GMG = G$. So $M^\# = G$.

Theorem: 2 Let $M = \begin{bmatrix} AB & B \\ C & 0 \end{bmatrix} \in K^{2l \times 2l}$, and $rank B \leq rank C, AB = BA$, then:

(1) $M^\#$ exists if and only if $rank B = rank C = rank(BC) = rank(CB)$

(2) If $M^\#$ exists, then

$$M^\# = \begin{bmatrix} B(CB)^\# A - B(CB)^\# AB(CB)^\# B & B(CB)^\# \\ F & -(CB)(CB)^\# AB(CB)^\# \end{bmatrix}$$

$$F = -(CB)(CB)^\# AB(CB)^\# A + (CB)^\# B - (CB)^\# [(CB)^\# AB] (CB)^\# B$$

Proof:

(1) Because $AB = BA$, we can obtain

$$\begin{bmatrix} AB & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} BA & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix} \begin{bmatrix} AB & C \\ B & 0 \end{bmatrix} \begin{bmatrix} A & I \\ I & 0 \end{bmatrix} = PNP^{-1}$$

So, $P = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix}$, $N = \begin{bmatrix} AB & C \\ B & 0 \end{bmatrix}$.

So $M^\#$ exists if and only if $N^\#$ exists. According to Theorem 1, $\begin{bmatrix} AB & C \\ B & 0 \end{bmatrix}^\#$ exists if and only

If $rankB = rankC = rank(BC) = rank(CB)$. So $M^\#$ exists if and only if

$$rankB = rankC = rank(BC) = rank(CB).$$

(2) If $M^\#$ exists, then $M^\# = PN^\#P^{-1} = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix} \begin{bmatrix} AB & C \\ B & 0 \end{bmatrix} \begin{bmatrix} A & I \\ I & 0 \end{bmatrix}$

According to Theorem 1, $N^\# = \begin{bmatrix} (BC)^\tau AC(BC)^\# & (BC)^\# - (BC)^\tau [AC(BC)^\#]^\flat B \\ C(BC)^\# & -C(BC)^\# AC(BC)^\# B \end{bmatrix}$.

It is easy to compute $M^\# = \begin{bmatrix} B(CB)^\# A - B(CB)^\# AB(CB)^\# B & B(CB)^\# \\ F & -(CB)(CB)^\# AB(CB)^\# \end{bmatrix}$

$$F = -(CB)(CB)^\# AB(CB)^\# A + (CB)^\# B - (CB)[(CB)^\# AB]^\flat (CB)^\# B$$

Let $A = I$, then $M = \begin{bmatrix} C & B \\ C & 0 \end{bmatrix}$, according to theorem 1, we can obtain the following corollary.

Corollary: 1 Let $M = \begin{bmatrix} A & B \\ A & 0 \end{bmatrix} \in K^{2l \times 2l}$, and $rankB \geq rankC$, then:

(1) $M^\#$ exists if and only if $rankB = rankA = rank(BA) = rank(AB)$

(2) If $M^\#$ exists, then

$$M^\# = \begin{bmatrix} (BA)^\tau A(BA)^\# & (BA)^\# - (BA)^\tau [A(BA)^\#]^\flat B \\ A(BA)^\# & -A(BA)^\# A(BA)^\# B \end{bmatrix}$$

Let $A = I$, then $M = \begin{bmatrix} B & B \\ C & 0 \end{bmatrix}$, according to theorem 2, we can obtain the following corollary.

Corollary: 2 Let $M = \begin{bmatrix} A & A \\ B & 0 \end{bmatrix} \in K^{2l \times 2l}$, and $rankA \leq rankB$, then:

(1) $M^\#$ exists if and only if $rankA = rankB = rank(AB) = rank(CA)$

(2) If $M^\#$ exists, then

$$M^\# = \begin{bmatrix} A(BA)^\# - A(BA)^\# A(BA)^\# A & A(BA)^\# \\ F & -(BA)(BA)^\# A(BA)^\# \end{bmatrix}$$

$$F = -(BA)(BA)^\# A(BA)^\# + (BA)^\# A - (BA)[(BA)^\# A]^\flat (BA)^\# A$$

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Source of Support: Nil, Conflict of interest: None Declared