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SOME RESULTS ON THE GROUP INVERSE OF BLOCK MATRIX OVER SKEW FIELDS

Junqing Wang* & Xin Fan

Department of Mathematics, Dezhou University, Shandong 253055, PR China.

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ABSTRACT

In this paper, we give the existences and the representations of the group inverse for block matrix $\begin{bmatrix} AC & B \\ C & 0 \end{bmatrix}$ *and* $\begin{bmatrix} AB & B \\ C & 0 \end{bmatrix}$ *under the special condition.*

Keywords: *block matrix; skew fields; group inverse.*

1. INTRODUCTION

Let K is a skew fields, $K^{m\times n}$ represents the set of $m\times n$ matrices over K. For $A\in K^{m\times n}$, if X satisfying $AXA = X$, $XAX = X$, $AX = XA$. We call X is the group inverse A, we write $X = A^{\#}$. In this paper, $A^{\pi} = I - AA^{\#}$.

Some authors and experts have given the representations of the group inverses in [1] and [2]. Actually, generalized inverses have wide applications in many areas such as special matrix theory, singular differential and difference equations and graph theory. The presentations of block matrix $\begin{bmatrix} AC & B \\ C & D \end{bmatrix}$ is givens in [3] under the assumption that A and $I - CA^{-1}B$ are both invertible over any fields; however, the representation of the group inverse $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is not given. In this paper, we obtain the representation of the block matrix $\begin{bmatrix} AC & B \\ C & 0 \end{bmatrix}$ and $\begin{bmatrix} AB & B \\ C & 0 \end{bmatrix}$.

2. PREPARATION KNOWLEDGE

In order to prove the main results, we give some lemmas.

Lemma: 1[1] Let $A\in K^{m\times n}$, then $A^{\#}$ exists if and only if $rankA = rankA^2$.

Lemma: 2[1] Let $A, B\in K^{m\times n}$, if $rankA = rankB = rank(AB) = rank(BA)$, 则, $(AB)^{\#}$ and $(BA)^{\#}$ exists, then the following conclusions hold:

(1) $AB(AB)^{\#}A = A$

(2) $A(BA)^{\#}BA = A$

(3) $BA(BA)^{\#}B = B$

(4) $B(AB)^{\#}A = BA(BA)^{\#}$

(5) $A(BA)^{\#} = (AB)^{\#}A$

***Corresponding author: Junqing Wang**
Department of Mathematics, Dezhou University, Shandong 253055, PR China.*

Lemma: 3[2] Let $A, X \in K^{n\times n}$, then $X = A^{\#}$ if and only if $AXA = A, AX = XA, rankX \leq rankA$.

3. MAIN CONCLUSION

Theorem: 1 Let $M = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} \in K^{2l\times 2l}$, and $rankB \geq rankC$, then：

(1) $M^{\#}$ exists if and only if $rankB = rankC = rank(BC) = rank(CB)$

(2) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} (BC)^{\pi} AC(BC)^{\#} & (BC)^{\#} - (BC)^{\pi}\left[AC(BC)^{\#}\right]^2 B \\ C(BC)^{\#} & -C(BC)^{\#} AC(BC)^{\#} B \end{bmatrix}$$

Proof: Because

$$rankM = rank\begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = rank\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = rankB + rankC$$

$$rankM^2 = rank\begin{bmatrix} ACAC + BC & ACB \\ CAC & CB \end{bmatrix} = rank\begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix}$$

First, we prove the 'if' part

According to the lemma 2, if $rankB = rankC = rank(BC) = rank(CB)$, then $CB(CB)^{\#} C = C$,

So $CAC = CB(CB)^{\#} CAC$, we get

$$rank\begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix} = rank\begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix} = rank(BC) + rank(CB)$$

So， $rankM = rankB + rankC = rankBC + rankCB = rankM^2$.

Second, we prove 'only if' part

Science $M^{\#}$ exists if and only if $rankM = rankM^2$, we have

$$rankB + rankC = rank\begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix} = rank\left(\begin{bmatrix} BC & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} I & 0 \\ AC & B \end{bmatrix}\right)$$

$$\leq \begin{pmatrix} BC & 0 \\ 0 & C \end{pmatrix} = rank(BC) + rankC$$

$$rankB + rankC = rank\begin{bmatrix} BC & 0 \\ CAC & CB \end{bmatrix} = rank\left(\begin{bmatrix} B & 0 \\ AC & I \end{bmatrix}\begin{bmatrix} C & 0 \\ 0 & CB \end{bmatrix}\right)$$

$$\leq \begin{pmatrix} C & 0 \\ 0 & CB \end{pmatrix} = rankC + rank(BC)$$

Then $rankB \leq rank(BC), rankB \leq rank(CB)$, then

$$rankB = rank(BC) = rank(CB)$$

Because $rankB \geq rank(C)$, then $rankB = rankC = rank(BC) = rank(CB)$.

(2) If $M^{\#}$ exists, $G = \begin{bmatrix} (BC)^{\pi} AC(BC)^{\#} & (BC)^{\#} - (BC)^{\pi}\left[AC(BC)^{\#}\right]^{2} B \\ C(BC)^{\#} & -C(BC)^{\#} AC(BC)^{\#} B \end{bmatrix}$

Write $G = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$, we can obtain,

$$MG = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} ACX + BZ & ACY + BW \\ CX & CY \end{bmatrix}$$

$$GM = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}\begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} XAC + YC & XB \\ ZAC + WC & ZB \end{bmatrix}$$

$$(MG)_{11} = (GM)_{11} = BC(BC)^{\#} \quad (MG)_{12} = (GM)_{12} = (BC)^{\pi} AC(BC)^{\#} B$$

$$(MG)_{21} = (GM)_{21} = 0 \qquad (MG)_{22} = (GM)_{22} = C(BC)^{\#} B$$

Then $MG = GM = \begin{bmatrix} BC(BC)^{\#} & (BC)^{\pi} AC(BC)^{\#} B \\ 0 & C(BC)^{\#} B \end{bmatrix}$

Because lemma 2, we can obtain:

$$MGM = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}\begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix}\begin{bmatrix} BC(BC)^{\#} & (BC)^{\pi} AC(BC)^{\#} B \\ 0 & C(BC)^{\#} B \end{bmatrix}$$

$$= \begin{bmatrix} ACBC(BC)^{\#} & AC(BC)^{\pi} AC(BC)^{\#} B + BC(BC)^{\#} B \\ CBC(BC)^{\#} & C(BC)^{\pi} AC(BC)^{\#} B \end{bmatrix} = \begin{bmatrix} AC & B \\ C & 0 \end{bmatrix} = M .$$

According to the same theory, $GMG = G.$ So $M^{\#} = G$.

Theorem: 2 Let $M = \begin{bmatrix} AB & B \\ C & 0 \end{bmatrix} \in K^{2l\times 2l}$, and $rankB \le rankC, AB = BA,$ then:

(1) $M^{\#}$ exists if and only if $rankB = rankC = rank(BC) = rank(CB)$

(2) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} B(CB)^{\#} A - B(CB)^{\#} AB(CB)^{\#} B & B(CB)^{\#} \\ F & -(CB)(CB)^{\#} AB(CB)^{\#} \end{bmatrix}$$

$$F = -(CB)(CB)^{\#} AB(CB)^{\#} A + (CB)^{\#} B - (CB)\left[(CB)^{\#} AB\right]^{2} (CB)^{\#} B$$

Proof:

(1) Because $AB = BA,$ we can obtain

$$\begin{bmatrix} AB & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} BA & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix}\begin{bmatrix} AB & C \\ B & 0 \end{bmatrix}\begin{bmatrix} A & I \\ I & 0 \end{bmatrix} = PNP^{-1}$$

So, $P = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix}$, $N = \begin{bmatrix} AB & C \\ B & 0 \end{bmatrix}$.

So $M^{\#}$ exists if and only if $N^{\#}$ exists. According to Theorem 1, $\begin{bmatrix} AB & C \\ B & 0 \end{bmatrix}^{\#}$ exists if and only If $rankB = rankC = rank(BC) = rank(CB)$. So $M^{\#}$ exists if and only if

$rankB = rankC = rank(BC) = rank(CB)$.

(2) If $M^{\#}$ exists, then $M^{\#} = PN^{\#}P^{-1} = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix}\begin{bmatrix} AB & C \\ B & 0 \end{bmatrix}\begin{bmatrix} A & I \\ I & 0 \end{bmatrix}$

According to Theorem 1, $N^{\#} = \begin{bmatrix} (BC)^{\pi} AC(BC)^{\#} & (BC)^{\#} - (BC)^{\pi}\left[AC(BC)^{\#}\right]^2 B \\ C(BC)^{\#} & -C(BC)^{\#}AC(BC)^{\#}B \end{bmatrix}$.

It is easy to compute $M^{\#} = \begin{bmatrix} B(CB)^{\#}A - B(CB)^{\#}AB(CB)^{\#}B & B(CB)^{\#} \\ F & -(CB)(CB)^{\#}AB(CB)^{\#} \end{bmatrix}$

$$F = -(CB)(CB)^{\#}AB(CB)^{\#}A + (CB)^{\#}B - (CB)\left[(CB)^{\#}AB\right]^2(CB)^{\#}B$$

Let $A = I$,then $M = \begin{bmatrix} C & B \\ C & 0 \end{bmatrix}$, according to theorem 1, we can obtain the following corollary.

Corollary: 1 Let $M = \begin{bmatrix} A & B \\ A & 0 \end{bmatrix} \in K^{2l\times 2l}$, and $rankB \geq rankC$, then:

(1) $M^{\#}$ exists if and only if $rankB = rankA = rank(BA) = rank(AB)$

(2) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} (BA)^{\pi} A(BA)^{\#} & (BA)^{\#} - (BA)^{\pi}\left[A(BA)^{\#}\right]^2 B \\ A(BA)^{\#} & -A(BA)^{\#}A(BA)^{\#}B \end{bmatrix}$$

Let $A = I$, then $M = \begin{bmatrix} B & B \\ C & 0 \end{bmatrix}$, according to theorem 2, we can obtain the following corollary.

Corollary: 2 Let $M = \begin{bmatrix} A & A \\ B & 0 \end{bmatrix} \in K^{2l\times 2l}$, and $rankA \leq rankB,$, then:

(1) $M^{\#}$ exists if and only if $rankA = rankB = rank(AB) = rank(CA)$

(2) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} A(BA)^{\#} - A(BA)^{\#}A(BA)^{\#}A & A(BA)^{\#} \\ F & -(BA)(BA)^{\#}A(BA)^{\#} \end{bmatrix}$$

$$F = -(BA)(BA)^{\#}A(BA)^{\#} + (BA)^{\#}A - (BA)\left[(BA)^{\#}A\right]^2(BA)^{\#}A$$

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