International Research Journal of Pure Algebra -3(12), 2013, 357-361
Available online through www.rjpa.info ISSN 2248–9037

MORE ON PAIRWISE ALMOST NORMAL SPACES

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(Received on: 27-11-13; Revised & Accepted on: 11-12-13)

ABSTRACT

The main focus of this paper is to introduce the properties of pairwise almost Hausdorffand pairwise almost normalspaces. Also we introduce the Urysohn lemma using pairwise almost normal.

Keywords: *Pairwise almost Hausdorff, pairwise $T_{\frac{1}{2}}$, pairwise almost normal, $\tau_1\tau_2$- R_0 space, $\tau_1\tau_2$- CL space.*

AMS Subject Classification: *54E55.*

1. INTRODUCTION

Since the formal study of bitopological space began with the paper of Kelly (1963) considerable effort has been expanded in obtaining appropriate generalization of standard topological properties in the bitopological category.

Some of the problems of and alternative definition for, bitopological compactness have been discussed by Cooke and Reilly (1975).

The class of nearly compact spaces was introduced by Singal & Mathur [11] and has since been considered by several authors, see [7, 8, 13].

In [2] the authors, introduced the notion of pairwise extremally disconnected spaces and investigated its fundamental properties.

The notion of R_0 topological spaces introduced by Shanin in 1943. Later, A. S. Davis [6] rediscovered it and studied some properties of this weak separation axiom. Several topologists further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, A. S. Davis alsointroduced the notion of R_1 topological space which is independent of bothT_0 and T_1 but strictly weaker than T_2.Bitopological forms of these concepts have appeared in the definitions of pairwise R_0 and pairwise R_1 spaces given by Mrsevic [14].

The main focus of this paper is to introduce the properties of pairwise almost Hausdorff and pairwise almost normal spaces .Also we introduce the Urysohn lemma using pairwise almost normal.

2. PRELIMINARIES

If A is a subset of X with a topology τ, then the closure of A is denoted by τ - cl (A) or cl (A), the interior of A is denoted by τ - int (A) or int (A) and the complement of A in X is denoted by A^c. Now we shall require the following known definitions and prerequisites.

Definition: 2.1 A topological space (X, τ) is **almost normal** if for each pair of disjoint sets A & B one of which is closed and other is regularly closed, there exists open sets U & V such that $A \subset U$, $B \subset V$ & $U \cap V = \phi$.

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Definition: 2.2 A subset A of a topological space (X, τ) is said to be **regular open** if A = int [cl (A)].

Definition: 2.3 A space X is **nearly compact** if every regularly open cover has a finitesubcover.

Definition: 2.4 Let f: $X\to \mathbb{R}$. f is **upper semi continuous** if $\{x: f(x) < b\}$ $\forall$ $b \in \mathbb{R}$ is open in X. f is **lower semi continuous** if $\{x: f(x) > b\}$ $\forall$ $b \in \mathbb{R}$.

f is continuous $\Leftrightarrow$ f is upper semi continuous and lower semi continuous.

Definition: 2.5 A subset A of (X, τ_1, τ_2) is said to be (τ_2, τ_1) - regularly closed if A = τ_2- cl (τ_1- int (A)).

Definition: 2.6 A bitopological space (X, τ_1, τ_2) is **pairwise almost Hausdorff**if for every pair x, y with $x \neq y$, $\exists$ a τ_1 - regular open neighborhood U of x and$\exists$ a τ_2 - regular open neighborhood V of y such that $U \cap V = \phi$.

Definition: 2.7 - A bitopological space (X, τ_1, τ_2) is called **pairwise Urysohn,** if for any two points x and y of X such that $x\neq y$, there exists a τ_i- open set U and a τ_j- open set V suchthat $x \in U$, $y \in V$, τ_j- cl (U) $\cap\tau_i$- cl (V) = ϕ where i, j = 1, 2 and $i \neq j$.

Definition: 2.8 - A bitopological space (X, τ_1, τ_2) is called **pairwise T_2** or **pairwise Hausdorff**if given distinct points x, y of X, there is a τ_i- open set U and a τ_j- open set V such that $x \in U$, $y \in V$, $U \cap V = \phi$where i, j = 1, 2 and $i \neq j$.

Definition 2.9: A space (X, τ) is said to be extremally disconnected if the closure of every open set is open.

Definition 2.10 [2]: A bitopological space (X, τ_1, τ_2) is said to be
i) **(τ_i, τ_j) - extremally disconnected** if τ_j- closure of every τ_i - open set is τ_j- open in (X, τ_1, τ_2).
ii) **Pairwise extremally disconnected** if (X, τ_1, τ_2) is (τ_i, τ_j) - extremally disconnected and (τ_j, τ_i) - extremally disconnected

3. PAIRWISE ALMOST NORMAL

Definition3.1: (X, τ_1, τ_2) is **pairwise $T_{\frac{1}{2}}$** if for every pair x, y with $x \neq y$, $\exists$ a τ_1 - open set U such that $x \in U$, $y\notin U$ or $\exists$ a τ_2 - open set V such that $y \in V$, $x\notin V$.

Theorem 3.1: (X, τ_1, τ_2) is pairwise $T_{\frac{1}{2}}$ if either (X, τ_1) is T_1 or (X, τ_2) is T_1.

Proof: Suppose that (X, τ_1) is T_1.

Let $x \neq y$ in X.

$\Rightarrow\exists$aτ_1 - open set U such that $x \in U$, $y\notin U$.

Suppose that (X, τ_2) is T_1.

$\Rightarrow\exists$aτ_2 - open set V such that $y \in V$, $x\notin V$.

Hence (X, τ_1, τ_2) is pairwise $T_{\frac{1}{2}}$.

Note 3.1:τ = {all open sets}
τ^* = {all regularly open sets}
$\tau^* \subset \tau$.

Definition 3.2: A bitopological space (X, τ_1, τ_2) is said to be an **$\tau_1\tau_2$ - R_0** space if each τ_1 - open set is τ_2 - regularly open.

Definition 3.3: A bitopological space (X, τ_1, τ_2) is said to be an **$\tau_1\tau_2$ - CL space** if each τ_1 - open set is τ_2 - closed.

Theorem 3.2: If X is an $\tau_1\tau_2$- R_0 space $\Leftrightarrow$ it is a $\tau_1\tau_2$- CL space.

Proof: If X is $\tau_1\tau_2$- CL space then it is $\tau_1\tau_2$- R_0 space since each τ_1 - open and τ_2 - closed set is $\tau_1\tau_2$- regularly open.

Conversely,

Let G $\in\tau_1$.

Then
A = X – (τ_2- cl (G) – G)
= X – τ_2- cl (G) $\cup$ G

Which is τ_1 - open and τ_2 - regular open.

Theorem 3.3: Every τ_1 - nearly compact subset of a pairwise almost Hausdorff space (X, τ_1, τ_2) is τ_2 - regularly closed.

Theorem 3.4: Let (X, τ_1, τ_2) is pairwise almost Hausdorff space. If (X, τ_1) is nearly compact then every τ_1 - regularly closed set is τ_2 - regularly closed.

$\Rightarrow \tau_1^* \subset \tau_2^*$, where τ_1^* = {τ_1 - regularly closed sets}.

Consequently, $\tau_1 \subset \tau_2$.

Proof: By hypothesis, (X, τ_1) is nearly compact.

$\Rightarrow$ every τ_1 - regularly closed subset A is τ_1- nearly compact.

But (X, τ_1, τ_2) is pairwise almost Hausdorff space.

Hence by the theorem 3.3, A is τ_2 - regularly closed.

$\Rightarrow$ every τ_1 - closed subset is τ_2 -closed (1)

Let G$\in \tau_1^*$.

Then A = X – G is τ_1^* - closed.

By (1) A is τ_2^* - regularly closed.

Hence G$\in \tau_2^*$.

$\Rightarrow \tau_1^* \subset \tau_2^*$.

Theorem 3.5: Let (X, τ_1^*, τ_2^*) be a nearly bi compact space that is, (X, τ_1^*) and $(X, \tau_2^*$) are nearly compact. Suppose that (X, τ_1^*, τ_2^*) is pairwise almost Hausdorff . Then $\tau_1^* = \tau_2^*$.

Proof: By hypothesis (X, τ_1^* , τ_2^*) is pairwise almost Hausdorff.

Suppose that (X, τ_1^*) is nearly compact.

By theorem 3.4, we have

$$\tau_1^* \subset \tau_2^* \quad (1)$$

Similarly,

$$\tau_2^* \subset \tau_1^* \quad (2)$$

From above (1) and (2), it follows that

$$\tau_1^* = \tau_2^* .$$

Theorem 3.6: Every pairwise almost Hausdorff space is pairwise Hausdorff.

Definition 3.4: (X, τ_1, τ_2) is **pairwise almost normal** if for a τ_1 - closed set and disjoint (τ_2, τ_1) - regularly closed set B, $\exists\tau_2$ - open nhd U of A and $\exists$ a τ_1 - open nhd V of B such that U $\cap$ V = ϕ.

Theorem 3.7: (Analogue of Urysohn Lemma) A bitopological space (X, τ_1, τ_2) is pairwise almost normal if and only if for each τ_1 - closed set A and (τ_2, τ_1) - regularly closed set B with $A \cap B = \phi$, $\exists$ a real valued function f on X such that $f(B) = \{0\}$, $f(A) = \{1\}$, $f(X) \subset [0, 1]$ and f is τ_1 - upper semi continuous and τ_2 - lower semi continuous.

Proof:
Step 1: (Sufficiency)

Suppose that A is (τ_2, τ_1) - regularly closed set and B be a τ_1 - closed set with $A \cap B = \phi$.

Take $f: X \rightarrow [0, 1]$ as the Urysohn function.

By the semi continuity of g, we have
i) $U = \{x \in X: f(x) < \frac{1}{2}\}$ is τ_2 - open.
ii) $V = \{x \in X: f(x) > \frac{1}{2}\}$ is τ_1 - open.

Also $U \cap V = \phi$.

Since $f(B) = \{1\}$, $B \subset V$.

Hence (X, τ_1, τ_2) is pairwise almost normal.

Step 2: Necessity

Suppose that (X, τ_1, τ_2) is an pairwise almost normal.

Let $B \subset X$ such that B is (τ_2, τ_1) - regularly closed.

Let $A \subset X$ such that A is τ_1 - closed set with $B \cap A = \phi$.

Put $B_0 = B$ and $K_1 = X - A$.

Then B_0 is (τ_2, τ_1) - regularly closed and K_1 is τ_1 - open and $B_0 \subset K_1$.

But (X, τ_1, τ_2) is pairwise almost normal.

Hence $\exists$ a τ_1 - open set $K_{\frac{1}{2}}$ and a (τ_2, τ_1) - regularly closed set $B_{\frac{1}{2}}$ such that

$$B_0 \subset K_{\frac{1}{2}} \subset B_{\frac{1}{2}} \subset K_1.$$

Apply our hypothesis to the pair B_0 ,$K_{\frac{1}{2}}$ and the pair $B_{\frac{1}{2}}$, K_1.

We obtain a τ_1 - open sets $K_{\frac{1}{2}}$, $K_{\frac{1}{4}}$ and (τ_2, τ_1) - regularly closed sets $B_{\frac{1}{4}}$, $B_{\frac{3}{4}}$ such that

$$B_0 \subset K_{\frac{1}{4}} \subset B_{\frac{1}{4}} \subset K_{\frac{1}{2}} \subset B_{\frac{1}{2}} \subset K_{\frac{3}{4}} \subset B_{\frac{3}{4}} \subset K_1.$$

Continue this process, we have a collection $\{B_s\}$ and another collection $\{K_s\}$ for $s = \frac{p}{2^q}$ with $p = 1, \ldots, 2^{q-1}$, and $q = 1, 2, \ldots \ldots$.

Take $K_s = \phi$ for $s \leq 0$ and $K_s = X$ if $s > 1$.

Also $B_s = \phi$ if $s < 0$, $B_s = X$ for $S \geq 1$ whenever s is any other dyadic fraction.

Then

$$K_r \subset K_s \subset B_s \subset B_t \quad \text{whenever } k \leq s \leq t.$$

Also, $B_s \subset K_t$ with $s < t$.

Define a function $f: X \rightarrow [0, 1]$ by $f(x) = \inf\{t: x \in K_t\}$ for all $x \in X$.

$\Rightarrow f(x) = \inf\{t: x \in B_t\}$ for all $x \in X$.

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$\Rightarrow 0 \leq f(x) \leq 1 \; \forall \; x \in X$.

Also, $f(x) = 0 \; \forall \; x \in B$ and $f(x) = 1 \; \forall \; x \in X - K = A$.

Furthermore, f is τ_1 - upper semi continuous and τ_2 - lower semi continuous.

This completes the proof.

Theorem 3.8: Every pairwise extremally disconnected & pairwise T_2 - space is pairwise Urysohn.

Proof: Suppose (X, τ_1, τ_2) be a pairwise extremally disconnected & pairwise T_2 - space.

To prove that (X, τ_1, τ_2) is pairwise Urysohn.

Let $x \neq y$ in (X, τ_1, τ_2).

Since (X, τ_1, τ_2) is pairwise Hausdorff, $\exists$ an τ_1 - open neighborhood U of x & τ_2 - open neighborhood V of y such that $U \cap V = \phi$.

But (X, τ_1, τ_2) is pairwise extremally disconnected, τ_2 - cl (U) is τ_1 - open &τ_1 - cl (U) is τ_2 - open.

Put τ_2 - cl (U) = G &τ_1 - cl (V) = H.

Then G is τ_1 - open & H is τ_2 - open.

Thus, we have $x \neq y$.

$\Rightarrow \exists \tau_1$ - closed neighborhood G of x &τ_2 - closed neighborhood H of y such that $G \cap H = \phi$.

$\Rightarrow (X, \tau_1, \tau_2)$ is pairwise Urysohn.

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Source of Support: Nil, Conflict of interest: None Declared