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PRIMARY L-FUZZY IDEALS

K. Swamy*
Assistant Professor in Mathematics, Aditya Engineering College, Surampalem, Peddapuram-533437, Andhra Pradesh, India.

G. V. Ramana
Assistant Professor in Mathematics, Aditya Engineering College, Surampalem, Peddapuram-533437, Andhra Pradesh, India.

A. V. S. N. Murty
Professor in Mathematics, Aditya Engineering College, Surampalem, Peddapuram-533437, Andhra Pradesh, India.

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ABSTRACT

In this paper it is studied that the concepts of primary L-fuzzy ideal and primary L-fuzzy ideal belonging to a prime L-fuzzy ideal [1]. Also it is proved that every Prime L-Fuzzy Ideal is a Primary L-Fuzzy Ideal but the converse is not true. Throughout this paper, X stands for a commutative ring with identity and L stands for a complete distributive lattice.

Keywords: *L-Fuzzy Subset, L-Fuzzy Ideal, Prime L-Fuzzy Ideal, Primary L-Fuzzy Ideal.*

1. INTRODUCTION

The concept of fuzzy subset was studied by L. A. Zadeh [4] generalizing the idea of characteristic function. By a fuzzy subset of a set X, we mean any function from X into the closed interval [0,1]. In the paper [6], J. A. Goguen replaced the valuation set [0, 1], by a complete lattice attempting to make a generalized study of fuzzy set theory by studying L fuzzy sets (where L is a complete lattice). A. Rosenfeld in his paper [3], studied fuzzy groups. Wang-Jin Liu [8] and Zhang Yue and Peng Xingtu [10] studied Fuzzy ideals, Prime ideals, Maximal ideals on a ring.

2. PRELIMINARIES

Definition: 2.1 (L-Fuzzy set) A function $A: X \to L$ is called an L-Fuzzy subset of X. The set of all L-Fuzzy sets in X is denoted by F(X).

Definition: 2.2 [L-Fuzzy sub ring] An L-fuzzy subset A of a ring X is said to be an L-fuzzy subring of X if it satisfies the following
1) $A(x) \wedge A(y) \le A(x-y)$ for all $x, y \in X$
2) $A(x) \wedge A(y) \le A(xy)$ for all $x, y \in X$.

Definition: 2.3 [L-fuzzy Ideal] Let X=(X, +,.) be a ring. An L-fuzzy subset A of X is called an L-fuzzy ideal iff
1) $A(x) \wedge A(y) \le A(x-y)$ for all $x, y \in X$.
2) $A(x) \vee A(y) \le A(xy)$ for all $x, y \in X$.

Note: The set of all L-fuzzy ideals in X is denoted by I(X).

Definition: 2.4 [Level Set] Let A be an L-fuzzy set in X. For $t \in L$, we define $A_t = \{x \in X / A(x) \ge t\}$.

Here A_t is called the t-cut (or a level subset) of A.

****Corresponding author: K. Swamy****
Assistant Professor in Mathematics, Aditya Engineering College, Surampalem, Peddapuram-533437, Andhra Pradesh, India.

Proposition: 2.5 Let X be a ring and $A \in F(X)$, $A \neq 0$. Then $A \in I(X)$ iff the level subset A_t, for any $t \in L$ with $t \leq A(\theta)$ ($\neq 0$), where θ is the zero element in X, is an ideal of X.

Theorem: 2.6 Let N be an arbitrary ideal of a ring X. Then there exists $A \in I(X)$ and $t \in L$ such that $A_t = N$.

Definition: 2.7: Let A be an L-fuzzy sub ring of a ring X. Then the set

$X_A = \{x \in X / A(x) = A(\theta)\}$ is called a base set of A (Where θ is the zero element in X).

Proposition: 2.8 If $A \in I(X)$ then the base set X_A is an ideal of X.

Definition: 2.9 Let $A \in I(X)$. A is said to be a prime L-fuzzy ideal, if for $a, b \in X$,

$A(a.b) = A(\theta)$ implies $A(a) = A(\theta)$ or $A(b) = A(\theta)$ (where θ is the zero element of ring X).

Proposition: 2.10 Let P be a prime L-fuzzy ideal, and suppose that

$P(a_1a_2\ldots\ldots a_n) = P(\theta)$. Then for at least one value of i, we have $P(a_i) = P(\theta)$.

Proof: Let P be a prime L-fuzzy ideal.

Suppose that $P(a_1a_2\ldots\ldots a_n) = P(\theta)$ and $P(a_i) \neq P(\theta)$ for all i.

We shall obtain a contradiction.

We have $P(a_1(a_2\ldots\ldots a_n)) = P(\theta)$ and $P(a_1) \neq P(\theta)$.

Since P is prime L-fuzzy ideal, we have $P(a_2a_3\ldots\ldots a_n) = P(\theta)$.

Again $P(a_2(a_3\ldots\ldots a_n)) = P(\theta)$ and $P(a_2) \neq P(\theta)$.

Since P is prime L-fuzzy ideal, we have $P(a_3a_4\ldots\ldots a_n) = P(\theta)$.

Continuing this process, finally we get $P(a_n) = P(\theta)$.

This is a contradiction.

Hence $P(a_i) = P(\theta)$ for some i.

The following is the straight forward verification.

Proposition: 2.11 Let $f: X \rightarrow Y$ be an epimorphism of rings and P be a Prime L-fuzzy ideal of Y. Then $f^{-1}(P)$ is a Prime L-fuzzy ideal of X.

3. PRIMARY L-FUZZY IDEALS

Definition: 3.1 (Primary L-fuzzy ideal) Let A be an L-fuzzy ideal of X. Then A is called a Primary L-fuzzy ideal of X, if for $a, b \in X$, $A(ab) = A(\theta)$ and $A(a) \neq A(\theta)$ implies $A(b^n) = A(\theta)$ for some positive integer n.

Note: 3.2 In fact, A is primary L-fuzzy ideal means, its base set X_A is primary ideal of X.

The following theorem can be easily proved

Theorem: 3.3 Every prime L-fuzzy ideal is a primary L-fuzzy ideal.

The converse of the above theorem is not true. i.e., every primary L-fuzzy ideal of a ring X is not prime L-fuzzy ideal of X. For,

Clearly $I = \langle 4 \rangle = 4Z$, the set of all multiples of 4 is a primary ideal in the ring of integers Z but not a prime ideal. We now prove that the characteristic function χ of I is primary L-fuzzy ideal but not prime L-fuzzy ideal.

Let L= {0, 1} be a lattice. Define $\chi : Z \rightarrow L$ by

$$\chi(a) = \begin{cases} 1 \; if \; a \in I \\ 0 \; if \; a \notin I \end{cases}$$

Clearly, χ is an L-fuzzy subset of Z.

We now prove that χ is an L-fuzzy ideal of Z. For this we have to prove that

1) $\chi(a) \wedge \chi(b) \leq \chi(a-b)$
2) $\chi(a) \vee \chi(b) \leq \chi(ab)$, for all $a, b \in Z$

Here four cases arise

i) $a \in I, b \notin I$
ii) $a \notin I, b \in I$
iii) $a \in I, \; b \in I$
iv) $a \notin I, b \notin I$.

1) Each of the cases i) , ii) and iv) : $\chi(a) \wedge \chi(b) = 0$. Since $\chi(x) = 0$ or 1 for any $x \in X$, $\chi(a-b) \geq 0 = \chi(a) \wedge \chi(b)$.

Case iii): i.e., $a \in I$, $b \in I$: So, $a-b \in I$.

$$\chi(a) \wedge \chi(b) = 1 \wedge 1 = 1 = \chi(a-b).$$

2) Each of the cases i), ii) and iii): clearly $\chi(a) \vee \chi(b) = 1$.Since I is an ideal, $ab \in I$. So $\chi(ab) = 1 = \chi(a) \vee \chi(b)$.

Case iv): Clearly $\chi(a) = 0 = \chi(b)$ and hence $\chi(ab) \geq 0 = \chi(a) \vee \chi(b)$.

Thus χ is an L-fuzzy ideal of Z.

Now, we have to prove that χ is a primary L-fuzzy ideal.

Let $a, b \in Z$ and $\chi(ab) = \chi(0) = 1$. So, $ab \in I = \langle 4 \rangle$.

If $a \in I$ then it is clear.

If $a \notin I$, then $b^n \in I$ for some $n>0$ (since $I = \langle 4 \rangle$ is primary ideal in Z) and hence $\chi(b^n) = 1 = \chi(0)$.

Hence χ is a primary L-fuzzy ideal of Z. Now,

χ is not prime L-fuzzy ideal, for; $\chi(2.6) = 1$ but $\chi(2) \neq 1$ and $\chi(6) \neq 1$.

Theorem: 3.4 Let $A \in I(X)$ and $A(e) \neq A(\theta)$(where 'e' is the unit element of the ring X). The following three statements are equivalent.

1) A is primary L-fuzzy ideal.
2) X_A is a primary ideal.
3) Every Zero divisor in the residue class ring X/X_A is nilpotent.

Proof: 1⇔2: Assume that A is primary L-fuzzy ideal.

Let $ab \in X_A$ and $a \notin X_A$.

i.e., $A(ab) = A(\theta)$ and $A(a) \neq A(\theta)$.

Since A is primary L-fuzzy ideal, $A(b^n) = A(\theta)$ for some positive integer n.

$$\Rightarrow b^n \in X_A.$$

Hence X_A is primary ideal. This proves $1 \Rightarrow 2$.

Conversely, assume that (2) holds i.e. X_A is primary ideal.

Let $A(ab) = A(\theta)$ and $A(a) \neq A(\theta)$.

Then $ab \in X_A$ and $a \notin X_A$.

Since X_A is primary ideal, $b^n \in X_A$ (for some positive integer n)

i.e., $A(b^n) = A(\theta)$. Hence A is primary L – fuzzy ideal. This proves $2 \Rightarrow 1$. Thus $1 \Leftrightarrow 2$.

$2 \Leftrightarrow 3$: Assume that (2) holds i.e. X_A is primary ideal.

Let $x+X_A$ be a zero divisor in the residue class ring X/X_A.

Then there exists $y+X_A \neq 0$ (i.e., $y \notin X_A$)

such that $(x+X_A)(y+X_A) = 0 + X_A$

$\Rightarrow \quad xy+X_A = 0+X_A$

$\Rightarrow \quad xy \in X_A$.

Since $y \notin X_A$ and X_A is primary ideal, $x^n \in X_A$ for some positive integer n

$\Rightarrow \quad x^n+X_A = 0$

$\Rightarrow \quad (x+X_A)^n = 0$.

i.e. $x+X_A$ is a nilpotent element in X/X_A.

Therefore every zero divisor in the ring X/X_A is nilpotent.

Conversely assume that 3) holds.

Let $ab \in X_A$ and $a \notin X_A$.

i.e., $ab + X_A = 0 + X_A$ and $a+X_A \neq 0+X_A$.

i.e., $(a+X_A)(b+X_A) = 0$ and $a+X_A \neq 0$.

i.e., $b+X_A$ is a zero divisor in X/X_A.

By our assumption, $b+X_A$ is nilpotent.

Therefore $(b+X_A)^n = 0+X_A$ for some positive integer n .

$\Rightarrow b^n+X_A = 0+X_A$.

$\Rightarrow b^n \in X_A$.

Hence X_A is primary ideal. Thus $2 \Leftrightarrow 3$.

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